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SEPARATION OF LAPLACE'S EQUATION*

BY

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ABSTRACT

The following results are established in this paper:

(I)** For the Laplace equation $\Delta\theta = 0$ in curvilinear co-ordinates (u, v, w) in Euclidean space to be directly separable† into two equations, one for S and one for Z, when the solution is $\theta = R(u, v, w)S(u, v)Z(w)$ with fixed R, it is necessary and sufficient that the surfaces w = constant (1) be orthogonal to the surfaces u = constant, v = constant and (2) be parallel planes, planes with a common axis, concentric spheres, spheres tangent at a common point, or one of the two sets of spheres generated by the co-ordinate lines when bicircular co-ordinates are rotated about the line joining the poles or about its perpendicular bisector.

(II) We have $R = 1$ always and only in the first three cases, namely, when the surfaces w = constant are parallel planes, planes with a common axis, or concentric spheres.

(III) In these three cases, but only these, the wave equation separates in the sense RSZ, and hence, for the wave equation, $R = 1$ automatically.

(IV) For further separation of the equation found above for S, when $S = X(u)Y(v)$ so that the solution is now $RXYZ$, it is necessary and sufficient that the co-ordinates be toroidal, or such that the wave equation so separates, or any inversions of these.

(V) The co-ordinates where the wave equation so separates, that is, admits solutions $RX(u)Y(v)Z(w)$, are only the well-known cases where this happens with $R = 1$, namely, degenerate ellipsoidal or paraboloidal co-ordinates (but see Sec. 8.2).

(VI). In these cases, but only these, $R = 1$ for the Laplace equation too.

(VII) Co-ordinates for RSZ or $RXYZ$ separation of the Laplace equation have the group property under inversion.

(VIII) In all cases R can be found by inspection of the linear element.

INTRODUCTION

0.1. The problem. Separation of variables is one of the simplest and most frequently used methods of solving partial differential equations subject to given boundary conditions. In this method, with which we assume the reader to be familiar, the surface over which the boundary values are specified must coincide with one of the co-ordinate

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**The separate results are numbered for ease of reference.

†See Sec. 0.2.

surfaces, and hence any restriction on the co-ordinates which can be used is also a restriction on the physical situations to which the method can be applied. It is natural to inquire how we shall determine all co-ordinate systems in which a given equation can be solved by separation of variables. Such is the type of problem with which the present paper is concerned. In a later paper we give a brief history of the subject and a more general discussion of separation methods.

Partly because it includes the wave equation in problems of this type (cf. Sec. 4.3), and partly because the results are still unknown, the emphasis here is on separation of variables in Laplace's equation. We confine the discussion to three dimensions as the case of greatest practical interest. Also this loss of generality, which is believed to be inconsequential, enables us to put the derivation in a form readily followed by the general reader whose interests are not primarily mathematical. Such considerations have weight because the problems treated are of interest to physicists as well as mathematicians.

0.2. The manner of separation. Turning now to the question of the mode of separation, we observe that a more general form than the XYZ generally considered will achieve the objectives desired. Specifically we may assume the form $RXYZ$, where R is a single fixed function[1] of the three co-ordinates: $R = R(u, v, w)$. If X, Y and Z form a complete set [2] and R is known, the operations can be carried out with no increase in complexity. It might be thought that a partial differential equation would have to be solved to get R, but we shall see that this is not the case.

Assuming the form $RXYZ$ rather than XYZ, we have a slightly milder restriction on the solutions and we might expect, therefore, to find a larger class of permissible co-ordinates. Such is indeed the case; with toroidal co-ordinates, for example, the equation is known to separate in the extended sense but not in the restricted sense [11], and the same is true for the so-called Dupin cyclides [7].

A discussion of separation must consider the method of obtaining the solution, as well as its form. In most circumstances the solution is obtained by separating the equation in the literal sense; that is, we multiply through by a fixed function of the co-ordinates to obtain two sets of terms, one set involving w only, for example, and the other involving u and v only. The same procedure is then used for the u, v terms. In the present work we assume not only that the solution has the form $XYZR$, but also that the equation can be separated in this way. Such an assumption is restrictive; in particular we do not obtain the Dupin cyclides even though the solutions have the prescribed form. One of the categories of co-ordinates in [3] is also omitted. In a subsequent paper we consider the general case, with its relation to this and other special modes of obtaining the solutions.

I. Solutions $R(u, v, w)S(u, v)Z(w)$

SEPARATION OF THE EQUATION

1.0. Laplace's equation, and the second derivative terms. Consider the Laplace operator under a transformation of co-ordinates from (x, y, z) in Euclidean three-

[1]Feshbach has raised the question of co-ordinate systems for which the wave equation separates in this sense.

[2]This means that a sufficiently well-behaved, but otherwise arbitrary, function $f(x, y)$ can be expanded in the form $\Sigma\, A_{ab}X_aY_b$; similarly for other pairs (X, Z), (Y, Z).

dimensional space to curvilinear co-ordinates (u, v, w). Since the appearance of cross derivatives involving w makes separability impossible, we consider at first the case of orthogonal co-ordinates. It will be found later that the general case of oblique co-ordinates can be easily reduced to this one (Sec. 4.1). In the curvilinear co-ordinates (u, v, w), then, let the linear element be

$$ds^2 = f^2\,du^2 + g^2\,dv^2 + h^2\,dw^2 \tag{1.0}$$

where f, g, and h are functions of u, v, and w. Then [11]

$$\Delta\theta = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\,\theta_u\right) + \frac{\partial}{\partial v}\left(\frac{hf}{g}\,\theta_v\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\,\theta_w\right)\right]$$

which becomes, if $\theta = R(u, v, w)S(u, v)Z(w)$,

$$\begin{aligned}\Delta\theta = SZR\bigg[&\frac{1}{f^2}\frac{S_{uu}}{S} + \frac{1}{g^2}\frac{S_{vv}}{S} + \frac{1}{h^2}\frac{Z''}{Z}\\ &+ F_1\frac{S_u}{S} + G_1\frac{S_v}{S} + H_1\frac{Z'}{Z} + F_2 + G_2 + H_2\bigg],\end{aligned} \tag{1.1}$$

where

$$F_1 = \frac{2}{f^2}\frac{R_u}{R} + \frac{1}{fgh}\left(\frac{gh}{f}\right)_u \tag{1.2}$$

and similarly for G_1 and H_1. Also

$$F_2 = \frac{1}{f^2}\frac{R_{uu}}{R} + \frac{1}{fgh}\left(\frac{gh}{f}\right)_u\frac{R_u}{R} \tag{1.3}$$

and similarly for G_2 and H_2.

For separability of $\Delta^2\theta = 0$ in the form (1.1) we require that the function Z separate off into an ordinary differential equation. That is, there must exist a function $A^2(u, v, w)$ such that when $\Delta\theta$ is multiplied by A^2, only the variable w appears in the coefficient of Z''/Z and Z'/Z, and w does not appear in the coefficient of any other differentiated terms. Moreover $A^2(F_2 + G_2 + H_2)$ must break up into the sum of a function of w only and a function of u, v only.

Using the condition on the coefficients of the terms involving second derivatives of S and Z, and bearing in mind the definition (1.0) of f, g and h, we see that the linear element must have the form

$$ds^2 = A^2[F^2\,du^2 + G^2\,dv^2 + H^2\,dw^2], \tag{1.4}$$

where F, G are functions of u, v only and H is a function of w only.

1.1. Permissible changes of variable. If we replace w by a function of w, and the u, v co-ordinates by new co-ordinates which still do not involve w, we may put (1.4) in the form

$$ds^2 = A^2[F^2\,(du^2 + dv^2) + dw^2] \tag{1.5}$$

where F is a function of u and v alone. Here we have written A for the new value of A and u, v, w for the new values of u, v and w. In addition, we have used the fact that any element $F^2\,du^2 + G^2\,dv^2$ can be mapped conformally on a plane.

The w transformation is certainly permissible, as it amounts merely to a change of scale for the w co-ordinate. Hence it does not alter the geometrical configuration of the co-ordinate surfaces. When we say that a change of variable is *permissible*, in this connection, we mean that the geometric properties relevant to separation of Laplace's equation are essentially unchanged. Hence proof of the existence of a certain geometric property in the transformed system shows that the same property was present in the original system.

That the change of u, v co-ordinates is permissible in this sense follows from the fact that the new linear element will have the form (1.4). Hence the condition that the equation should separate is still satisfied (note that Eq. (1.8) will also persist in form). Moreover, since every linear element of the form (1.4) corresponds to a triply orthogonal system, we see that the equations used later, (2.0) and (2.1), will continue to be valid in the new system. From these and similar remarks concerning changes of co-ordinates we conclude finally that it suffices to specify the surfaces w = constant. If the equation separates in our sense for a particular system of u and v surfaces orthogonal to these it will separate for every other such system. Conversely, if a change of u and v co-ordinates leads to a certain set of u and v surfaces, and it is then found that the w surfaces must have certain special properties, these properties will in fact persist for every choice of the u or v surfaces. To simplify the analysis these two operations, changing the w scale and changing the u, v co-ordinates, will be repeatedly used in the ensuing discussion.

1.2. The first derivative terms. Turning now to the coefficients of terms involving first derivatives, we use the fact that AH_1 must be a function of w only to find, by virtue of (1.2) and (1.5), that

$$A^2\left[\frac{2}{A^2}\frac{R_w}{R} + \frac{1}{F^2A^3}(F^2A)_w\right] = h_1(w),$$

which simplifies to

$$\frac{2R_w}{R} + \frac{A_w}{A} = h_1(w).$$

We integrate with respect to w, noting that the constant of integration may be an arbitrary function of u and v, and we take the exponential of each side, to find finally

$$R^2A = h_2(w)F_3(u, v). \tag{1.6}$$

Since it is permitted that R involve u, v, and w we may always modify R in such a way that

$$R^2A = 1. \tag{1.7}$$

Thus, the term h_2 in (1.6) may be absorbed in Z and F_3 in S. Equation (1.7) leads to the result: *one may determine R explicitly by putting the linear element in the form* (1.5), *and then taking* $R = 1/A^{1/2}$.

It may be verified that the coefficients of S_u/S and S_v/S are already functions of u and v only, without any new condition (actually they are zero). From separation of the equation, therefore, we have only (1.5), (1.7), and an extra condition,

$$A^2(F_2 + G_2 + H_2) = h_4(w) + F_4(u, v). \tag{1.8}$$

It will be found that (1.8) is a consequence of the Euclidean character of the space.

USE OF PROPERTIES OF EUCLIDEAN SPACE

2.0. The functional form of A. If the space is Euclidean then the linear element (1.0) satisfies the relations [2], [6],

$$f_{uv} - \frac{g_w f_v}{g} - \frac{h_v f_w}{h} = 0, \tag{2.0}$$

$$\left[\frac{g_u}{f}\right]_u + \left[\frac{f_v}{g}\right]_v + \frac{f_w g_w}{h^2} = 0. \tag{2.1}$$

We also have those relations obtained by simultaneous cyclic permutation of (fgh) and (uvw). These equations, which state that the Riemannian curvature of the space is zero, give a necessary and sufficient condition that the co-ordinate system be imbedded in Euclidean space.

In terms of (1.5), the relation (2.0) containing g_{uw} becomes

$$\frac{A_{uw}}{A_u} = 2\frac{A_w}{A} \tag{2.2}$$

after simplification. Integrating with respect to w, as in the derivation of (1.6), we find that $(1/A)_u$ is a function of u and v alone. The same is true of $(1/A)_v$. Thus it follows that $1/A = t(u, v) + H(w)$, or

$$A = 1/[t(u, v) + H(w)]. \tag{2.3}$$

The two relations (2.0) used in the derivation are now satisfied, and in terms of the new linear element the third gives

$$t_{uv} = F_v t_u/F + F_u t_v/F. \tag{2.4}$$

Turning now to the relations (2.1), we find that (2.1) as it stands leads, without detailed calculation, to an equation of the form[3]

$$H'^2 + c_1 H^2 + 2c_2 H + c_3 = 0, \tag{2.5}$$

where the coefficients c are functions of u and v only. Differentiating with respect to w we find that

$$H'(H'' + c_1 H + c_2) = 0. \tag{2.6}$$

Since the solution depends on w only, while the coefficients depend on u and v only, we may give to u and v any constant values to get a linear differential equation for H with constant coefficients. This equation may be solved to give

$$H = 0,\ w,\ w^2,\ e^w,\ \sinh w,\ \cosh w,\ \sin w \tag{2.7}$$

as the only possible values of H that are essentially distinct. Here the case $H' = 0$ has been reduced to $H = 0$ by absorbing the constant value of H in the function t. If H is not zero we substitute back into (2.6) and find that the coefficients, which we already knew were independent of w, are actually constant. This result will be needed later.

In summary, we have found that the linear element must be of the form

$$ds^2 = \frac{F^2(du^2 + dv^2) + dw^2}{(t + H)^2} \tag{2.8}$$

[3]The equation is written explicitly in (2.9).

with t and F functions of u and v only, while H is one of the functions (2.7). The only additional conditions are (1.8), (2.1), and (2.4).

To use the relations (2.1) we substitute for f, g and h their values as given by (2.8). The equation (2.1) as it stands gives

$$\frac{t_{uu} + t_{vv}}{t + H} = \frac{t_u^2 + t_v^2 + F^2H'^2}{(t + H)^2} + \left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v \tag{2.9}$$

while the sum of the two others found by permutation leads to

$$t_{uu} + t_{vv} = 2\,\frac{t_u^2 + t_v^2 + F^2H'^2}{t + H} - 2F^2H'' \tag{2.10}$$

and their difference gives

$$t_{uu} - t_{vv} = 2t_u\frac{F_u}{F} - 2t_v\frac{F_v}{F}\,. \tag{2.11}$$

The system (2.9)-(2.11), slightly simpler than (2.1), is equivalent to it. These three equations and (2.4) give the necessary and sufficient condition that the metric (2.8) be for orthogonal co-ordinates in Euclidean space. The fact that ds^2 has the form (2.8), plus the relation (1.8), on the other hand, gives the necessary and sufficient condition that this co-ordinate system be one in which Laplace's equation separates. Systems satisfying both sets of conditions, and only those, are the ones we are seeking.

2.1. The second fundamental form. It is known from differential geometry [2], [6] that a triply orthogonal system is completely determined, except for its orientation in space, by the linear element ds^2. Hence the second fundamental form (as well as the first) for each of our co-ordinate surfaces is specified by f, g and h of Eq. (1.0) and we might expect to find specific relations from which they could be computed. Such relations actually exist [2], [7]; on the surface w = constant we have

$$-d\bar{x}\cdot d\bar{N} = l\,du^2 + m\,du\,dv + n\,dw^2,$$

where

$$l = -ff_w/h, \qquad m = 0, \qquad n = -gg_w/h; \tag{2.12}$$

the results in other cases are obtained by cyclic permutation of u, v, w and f, g, h.

In particular for the linear element (2.8) we obtain that

$$-d\bar{x}\cdot d\bar{N} = \frac{F^2H'}{(t + H)^2}\,(du^2 + dv^2). \tag{2.13}$$

On the other hand setting $dw^2 = 0$ in (2.8) gives

$$d\bar{x}\cdot d\bar{x} = \frac{F^2}{(t + H)^2}\,(du^2 + dv^2) \tag{2.14}$$

for the first fundamental form. Since the two forms (2.13) and (2.14) are proportional, we know that the surfaces w = constant must be spheres or planes [13]. Computing the Gaussian curvature as the ratio of the two discriminants d^2/D^2 (Ref. [13]) we find the radius of the sphere corresponding to a given value of w:

$$\text{radius} = 1/H'(w). \tag{2.15}$$

This result will frequently be used in the ensuing investigation.

SEPARATE EXAMINATION OF CASES

3.0. The case $H = 0$. For each value of H in (2.7) we obtain a new set of relations for F. These equations are not easy to solve as they stand, and our procedure will be to seek a change of u, v variables that will reduce them to simpler form. By the discussion of Sec. 1.1 we know that any restrictions on the w surfaces obtained after the change must have been valid before it as well.

In case $H = 0$, as assumed here, we see by (2.13) that $l = m = n = 0$ and hence the surfaces w = constant must be planes [13]. On one of these planes let us pick the u, v co-ordinates so that the linear element takes the Cartesian form $du^2 + dv^2$. This is possible, since the surface is a plane. By comparison with (2.8) when $dw^2 = 0$ we see that the present u and v co-ordinates make $F/t = 1$. Also since both t and F are independent of w, we now have $F/t = 1$ for all values of w, not merely for the constant value first selected.

With this procedure Eqs. (2.10) and (2.11) become

$$t_{uu} + t_{vv} = 2(t_u^2 + t_v^2)/t,$$

$$t_{uu} - t_{vv} = 2(t_u^2 - t_v^2)/t,$$

whence, after adding and dividing by t_u,

$$t_{uu}/t_u = 2t_u/t$$

with a similar result for t_v. Integrating twice we find that

$$1/t = uf_1(v) + f_2(v),$$

and a corresponding result with u and v interchanged. The two together show that $1/t$ must have the form $a + bu + cv + duv$, with a, b, c, d constant; Eq. (2.4) tells us that $d = 0$. The remaining conditions (2.9) and (1.8) are now satisfied, so that there is no other restriction.

If $b = c = 0$ the surfaces w = constant represent parallel planes, as we see by comparison with the linear element for Euclidean co-ordinates. If b or c is not zero, however, we have $ds^2 = du^2 + dv^2 + u^2\,dw^2$ or, renaming the variables, $ds^2 = dr^2 + dz^2 + r^2\,d\theta^2$. This shows that the w = constant surfaces are planes with a common axis.

3.1. The case $H = w$. Turning now to the case $H = w$ we find from (2.10) that

$$(t_{uu} + t_{vv})(t + w) = 2(t_u^2 + t_v^2 + F^2).$$

Since this is an identity in w the coefficient of w must vanish. Consequently $t_{uu} + t_{vv} = 0$, and hence also $t_u^2 + t_v^2 + F^2 = 0$. This is impossible since it makes $F = 0$.

3.2. The case $H = w^2$. Next if $H = w^2$ we find that

$$(F_u/F)_u + (F_v/F)_v = 0, \tag{3.1}$$

$$t_{uu} + t_{vv} = 4F^2, \tag{3.2}$$

$$t_{uu} + t_{vv} = (t_u^2 + t_v^2)/t, \tag{3.3}$$

by equating to zero the coefficients of w^4, w^2 and 1 in (2.9). Nothing new is obtained from (2.10), and hence these with (2.4) and (2.11) are the only conditions. Equations (3.1)

and (3.3) are equivalent to the statements that log F and log t are harmonic functions.

By simply writing the Gauss equation for curvature in terms of the first fundamental form, we find that for any F satisfying (3.1), this curvature vanishes for the surfaces with $ds^2 = F(du^2 + dv^2)$. These surfaces therefore are developable, and we may introduce a new set of u, v co-ordinates which will make $F = 1$. When this is the case, Eq. (2.4) tells us that $t_{uv} = 0$, so that $t = \alpha(u) + \beta(v)$. Equation (3.2) now reduces to $\alpha'' + \beta'' = 4$, which gives $\alpha'' = 2 + c$, $\beta'' = 2 - c$ with c a constant. It follows that $\alpha = u^2$, $\beta = v^2$, after making a change of variable, if necessary, to eliminate the arbitrary constants. Our linear element now takes the form

$$ds^2 = \frac{du^2 + dv^2 + dw^2}{(u^2 + v^2 + w^2)^2} \tag{3.4}$$

after t has been given its value as determined above. We observe from (7.1) and (7.2) that the linear element (3.4) is the one which would be obtained by inversion of Cartesian co-ordinates in the origin. Because the linear element is sufficient to determine the co-ordinate system completely, as noted in Sec. 2.1, it follows that the co-ordinate surfaces, as well as the linear element, will be the same as those which would be obtained by the inversion described. In particular the surfaces w = constant must consist of a plane and a set of spheres all tangent to it at one point. This result, incidentally, can be obtained directly from (3.4), as in the examples considered below.

3.3. The case $H = e^w$. Turning now to the case $H = e^w$ we substitute in (2.10) and equate to zero the coefficients of 1 and e^w, respectively, to find:

$$t(t_{uu} + t_{vv}) = 2(t_u^2 + t_v^2),$$

$$t_{uu} + t_{vv} = -2F^2t,$$

which gives

$$t_u^2 + t_v^2 + F^2t^2 = 0. \tag{3.5}$$

Equation (3.5) implies that t is zero. We have then from (2.8) that

$$ds^2 = \frac{F^2(du^2 + dv^2) + dw^2}{e^{2w}}. \tag{3.6}$$

The other relations are now all satisfied, if F is suitably restricted.

By the second fundamental form the radius of the sphere w = constant is e^{-w}. The distance from $w = \infty$ to the sphere along $u = a$, $v = b$ is also e^{-w}, by Eq. (3.6). Thus the spheres are concentric with $w = \infty$ as the common center.

3.4. The case $H = \sinh w$. When $H = \sinh w$, we substitute into (2.10) as usual, then put everything in terms of sinh w by using $\cosh^2 w = 1 + \sinh^2 w$, and finally equate to zero the coefficients of 1 and sinh w. We thereby obtain

$$t(t_{uu} + t_{vv}) = 2(t_u^2 + t_v^2 + F^2), \tag{3.7}$$

$$t_{uu} + t_{vv} = -2F^2t. \tag{3.8}$$

Together these relations show that

$$t_u^2 + t_v^2 + F^2 + F^2t^2 = 0 \tag{3.9}$$

which is impossible, since it makes $F = 0$.

3.5. The case $H = \cosh w$. If $H = \cosh w$ we proceed as above to obtain, from (2.9) and (2.10) after slight simplification,

$$t_{uu} + t_{vv} + 2tF^2 = 0, \tag{3.10}$$

$$t_u^2 + t_v^2F^2(t^2 - 1) = 0, \tag{3.11}$$

$$F^2 + (F_u/F)_u + (F_v/F)_v = 0. \tag{3.12}$$

Let us notice by (2.15) that the surface $w = 0$ is a plane, so that reasoning as before we may assume $F = t + 1$.

From (2.4) we have that

$$t_{uv} = 2t_ut_v/(t + 1)$$

which may be integrated, divided by $(f + 1)^2$ and integrated again, to give finally

$$1/(t + 1) = \alpha(u) + \beta(v). \tag{3.13}$$

After division by $(t + 1)^4$ Eq. (3.11) becomes

$$\alpha'^2 + \beta'^2 + 1 = 2(\alpha + \beta). \tag{3.14}$$

When differentiated with respect to u, v being constant, this leads to the linear equation

$$\alpha'(\alpha'' - 1) = 0.$$

We have a similar relation for β; whence we conclude that

$$\begin{aligned} \alpha &= a \qquad \text{or} \qquad (1/2)u^2 + au + b, \\ \beta &= A \qquad \text{or} \qquad (1/2)v^2 + Av + B. \end{aligned} \tag{3.15}$$

That both α and β cannot be constant is seen by (3.13), (3.10), and (3.11); the equations require $F = 0$, which is not permissible. If α alone is constant, moreover, the relation (3.14) tells us that $A^2 + 1 = 2(\alpha + \beta)$. The value of F thus obtained does not satisfy (3.12). Hence neither α nor β is constant. After substituting (3.15) into (3.14) to find $b + B = 1/2$, making a change of variable to get $a = A = 0$, and using (3.13), we find that

$$t = 2/(u^2 + v^2 + 1) - 1.$$

This value of t, which satisfies all relations, leads to

$$ds^2 = \frac{4(du^2 + dv^2) + (1 + u^2 + v^2)^2\, dw^2}{[2 + (\cosh w - 1)(1 + u^2 + v^2)]^2}$$

as the linear element when $H = \cosh w$. Using $\cosh w - 1 = 2 \sinh^2 (w/2)$, writing w for $w/2$, and taking $\cosh^2 w = 1 + \sinh^2 w$ in the denominator, we find that

$$ds^2 = \frac{du^2 + dv^2 + (1 + u^2 + v^2)^2\, dw^2}{[\cosh^2 w + (u^2 + v^2) \sinh^2 w]^2} \tag{3.16}$$

which assumes the proper form, we note, when $w = 0$.

The change of variable $u = r \cos \theta$, $v = r \sin \theta$ leads to

$$ds^2 = \frac{dr^2 + r^2\, d\theta^2 + (1 + r^2)^2\, dw^2}{[1 + (1 + r^2) \sinh^2 w]^2}$$

and in this case the surfaces $\theta =$ constant are all planes, as we see by computing the second fundamental form. On these surfaces $\theta =$ constant we have

$$ds^2 = \frac{dt^2 + dw^2}{[\sin^2 t + \sinh^2 w]^2}$$

after making the change of variable $r = \cot t$. The linear element last obtained coincides with that for bipolar co-ordinates, whence we conclude that the spheres $w =$ constant must be one of the sets of surfaces generated when bipolar co-ordinates are revolved about the line joining the two poles. The fact that $w = 0$ is plane shows that our spheres must be those which have their centers in the line joining the two poles.

3.6. The case $H = \sin w$. The final case is $H = \sin w$, which gives

$$t_u^2 + t_v^2 = F^2(t^2 - 1), \tag{3.17}$$

$$t_{uu} + t_{vv} = 2F^2 t, \tag{3.18}$$

$$(F_u/F)_u + (F_v/F)_v = F^2 \tag{3.19}$$

when we substitute in (2.9) and (2.10), replace $\cos^2 w$ by $1 - \sin^2 w$, and equate to zero the coefficients of 1, $\sin w$ and $\sin^2 w$. By (2.15) the surface $w = \pi/2$ is a plane. Hence we may assume $F = t + 1$ to obtain the result (3.13), as before. Equation (2.4) is now satisfied; the only ones remaining are (2.11) and those just obtained.

By (3.13) and (3.17)

$$\alpha'^2 + \beta'^2 + 2(\alpha + \beta) = 1.$$

Proceeding as in the discussion of (3.15) we find that neither α' nor β' is zero, and that

$$t = 2/(1 - u^2 - v^2) - 1$$

so that the linear element becomes

$$ds^2 = \frac{4(du^2 + dv^2) + (1 - u^2 - v^2)\, dw^2}{[2 + (\sin w - 1)(1 - u^2 - v^2)]^2}.$$

Continuing as for (3.16) we find that the surfaces $w =$ constant are the other spheres generated by the co-ordinate lines when bipolar co-ordinates are rotated, that is, a set of spheres passing through a single fixed circle. This result can also be reached directly by inspection of ds^2.

CONCLUDING REMARKS

4.0. Dismissal of an auxiliary condition. To complete the discussion we must show that (1.8) is satisfied in each of the cases considered. With this end in view we use (1.7), (1.5), and (1.3) to obtain, for the left member of (1.8),

$$\frac{1}{4F^2}\left[\frac{A_u^2 + A_v^2 + F^2 A_w^2}{A^2} - 2\,\frac{A_{uu} + A_{vv} + F^2 A_{ww}}{A}\right]. \tag{4.1}$$

Replacing A by its proper value (2.3) and simplifying we find

$$\frac{1}{4F^2}\left[2\,\frac{t_{uu}+t_{vv}+H''F^2}{t+H}-3\,\frac{t_u^2+t_v^2+F^2H'^2}{(t+H)^2}\right] \tag{4.2}$$

which becomes

$$(t_{uu}+t_{vv})/8F^2(t+H)-H''/4(t+H) \tag{4.3}$$

if we use (2.10).

When $H = 0$, the expression in (4.3) is a function of u and v alone, so that (1.8) is certainly satisfied. Similarly, if $H = w^2$ we may replace $t_{uu} + t_{vv}$ by its value (3.2), whence it is seen again that (4.3) has the proper form as prescribed by (1.8). If $H = e^w$ the result is again true, since t is zero; for $H = \cosh w$ it is a consequence of (3.10); for $H = \sin w$ it follows from (3.18). Thus Eq. (1.8) is satisfied automatically in all cases, and the co-ordinate systems hitherto obtained will actually lead to separation. What we have shown is that (1.8) is a consequence of the fact that the space is Euclidean.

4.1. Non-orthogonal co-ordinates. As noted, the w-surfaces are orthogonal to the others, since there must be no cross derivative terms involving the variables to be separated. It is really not necessary, however, that the u and v surfaces be orthogonal to each other, although up to now we have assumed this to be the case. Thus, orthogonality of the u and v surfaces is essential to our derivations, because the relations of Sec. 2 presuppose that the w = constant surfaces are imbedded, or at any rate can be imbedded, in a triply orthogonal system. Our aim now is to discard this condition of orthogonality as an initial assumption, and to show that the equations dependent thereon will be satisfied anyway as a consequence of separation.

To this end we assume that

$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2 + h\,dw^2 \tag{4.4}$$

rather than (1.0) and obtain [12] in place of (1.1),

$$\frac{\partial}{\partial u}\left(\frac{gh^{1/2}}{d}\,\theta_u-\frac{fh^{1/2}}{d}\,\theta_v\right)+\frac{\partial}{\partial v}\left(\frac{eh^{1/2}}{d}\,\theta_v-\frac{fh^{1/2}}{d}\,\theta_u\right)+\frac{\partial}{\partial w}\left(\frac{eg}{h^{1/2}d}\,\theta_w\right)=0. \tag{4.5}$$

Upon assuming a solution RSZ as in Sec. 1.0 we find that

$$\frac{g}{d^2}\,\frac{S_{uu}}{S}-2\,\frac{f}{d^2}\,\frac{S_{uv}}{S}+\frac{e}{d^2}\,\frac{S_{vv}}{S}+\cdots=0, \tag{4.6}$$

where the terms not written involve first derivatives of the unknown functions only, besides e, f, g, h and R. In (4.5) and (4.6) d^2 is the discriminant of the $u - v$ quadratic form,

$$d^2 = eg - f^2. \tag{4.7}$$

Since the equation separates, there exists a function $A(u, v, w)$ such that when the equation is multiplied by A, the coefficients of terms involving S are functions of u and v only, while the coefficients of terms with Z are functions of w only (cf. Sec. 1). Also the term free of unknowns must break up into a function of u, v plus a function of w. For our present purposes we need only the coefficients of S_{uu}, S_{uv} and S_{vv}, which tell us that

$$Ag/d^2,\ Af/d^2,\ Ae/d^2 \tag{4.8}$$

are functions of u and v alone. It follows that any combination of these expressions is also a function of u and v alone. Hence, in particular,

$$\left(\frac{Ag}{d^2}\right)\left(\frac{Ae}{d^2}\right) - \left(\frac{Af}{d^2}\right)^2 = \frac{A^2}{d^2} \tag{4.9}$$

has this property. Multiplying (4.8) by A and using (4.9) we find that e/A, f/A and g/A are functions of u and v alone, so that the linear element (4.4) has the form

$$ds^2 = A[e_1(u, v)\, du^2 + f_1(u, v)\, du\, dv + g_1(u, v)\, dv^2] + h dw^2. \tag{4.10}$$

Confining our attention to the terms in brackets, we see that it is always possible to make a change of variables, replacing u by $\bar{u}(u, v)$ and v by $\bar{v}(u, v)$, so that in the new variables we will have $f_1 = 0$. This is an analytic expression of the well-known geometrical fact that every surface admits a set of parametric curves which are orthogonal. When such a change of variables is made, the linear element (4.10) reduces to the form (1.0).

What we have shown is that there exists a change of u and v parameters which will make the new u and v surfaces orthogonal to each other, if they were not originally. Such a change is permissible in the sense of Sec. 1.1, and may therefore be carried out at the beginning of the investigation. We now have a linear element of the form (1.0), and the foregoing derivation proceeds without further change. Thus we have completed the proof of the first result:

With fixed R, if solutions $R(u, v, w)S(u, v)Z(w)$ satisfy Laplace's equation, and if separate differential equations for S and Z can be found by multiplying the equation by a suitably chosen function, then the co-ordinate surfaces w = constant must be orthogonal to the other two co-ordinate surfaces, and must consist of parallel planes, planes with a common line of intersection, spheres tangent at a common point, concentric spheres, the plane and set of spheres obtained when one set of bicircular co-ordinates is revolved about the line joining the poles, or a set of spheres all passing through a single fixed circle. Also, if the surfaces w = constant have any of these forms, and if the u and v surfaces are orthogonal to them, then Laplace's equation can always be separated in the prescribed manner.

4.2. Cases for which $R = 1$. Let us inquire when we may assume that $R = 1$. A necessary and sufficient condition is that R have the form $F(u, v)G(w)$, since in that case, but not otherwise, R may be absorbed in the solution $S(u, v)Z(w)$. From the equation $R^2A = 1$ we see that this condition, and hence the possibility of $R = 1$, is satisfied when, and only when, A also has the form $F(u, v)G(w)$. We know, however, that A must have the form (2.3), whence we conclude that t, H, or both must be constant for $R = 1$. These considerations lead to the second result:

We may assume that $R = 1$ always and only when the surfaces w = constant consist of parallel planes, planes with a common line of intersection, or concentric spheres.

4.3. The wave equation. Next let us consider the possibility of separating the wave equation. As noted, the theory is simpler than the preceding and included in it. Writing the equation in the form (*cf.* [16])

$$\Delta\theta + k\theta = 0, \tag{4.11}$$

we see that $F_2 + G_2 + H_2$ of (1.1) becomes $F_2 + G_2 + H_2 + k$, and this is the only

change. All results previously obtained are valid here too, then, except for (1.8), which becomes

$$A^2(F_2 + G_2 + H_2 + k) = F_5(u, v) + H_5(w). \tag{4.12}$$

But in the course of the foregoing investigation it was shown that (1.8), originally postulated independently as a result of the separation, is actually not an independent relation, but follows automatically from the others (Sec. 4.0). Though not required in itself for separation of the wave equation, therefore, this relation must nevertheless hold, in view of the other conditions. Combining it with (4.12) we see that

$$A^2 = F_6(u, v) + H_6(w). \tag{4.13}$$

If t of (2.3) is not constant it depends on u, say, and the same is true of F_6 in view of (1.8) and (4.13). Equating the two expressions for A, we differentiate with respect to u, solve for $(t + H)^3$, and differentiate the result with respect to w to get

$$3(t + H)^2H' = 0$$

which shows that H is constant. Thus either H or t must be constant. We therefore have $R = 1$, immediately, for the case of partial separation. In this way we obtain the third result:

The wave equation separates in the sense RSZ for the three cases having $R = 1$, and for these only.

II. Solutions $R(u, v, w)X(u)Y(v)Z(w)$

FURTHER SEPARATION OF THE EQUATION

5.0. General. It has been supposed hitherto that the solution is only partially separated, being of the form $R(u, v, w)S(u, v)Z(w)$ with a fixed function R, If we assume that the solution separates further to give the form $R(u, v, w)X(u)Y(v)Z(w)$, so that $S(u, v) = X(u)Y(v)$ for each function of the family, then we obtain new conditions on the co-ordinates. Because the cross derivative terms make separability impossible, for example, it is known at the outset that all three co-ordinate surfaces must now be orthogonal.

The initial stage of the separation led to an ordinary differential equation for $Z(w)$ and, with m constant, to

$$\frac{A^2}{f^2}\frac{S_{uu}}{S} + \frac{A^2}{g^2}\frac{S_{vv}}{S} + A^2F_1\frac{S_u}{S} + A^2G_1\frac{S_v}{S} + F_4(u, v) = m \tag{5.1}$$

for $S(u, v)$, as we see by using (1.8) and noting that each of the separated groups of terms must be constant. This is true because the sum must be zero, and one expression involves w only while the other involves u and v only. When S has the assumed form XY, (5.1) becomes

$$\frac{A^2}{f^2}\frac{X''}{X} + \frac{A^2}{g^2}\frac{Y''}{Y} + A^2F_1\frac{X'}{X} + A^2G_1\frac{Y'}{Y} + F_4 = m. \tag{5.2}$$

For separation there must be a function $J(u, v)$ such that (5.2) separates when multiplied by it. In particular the term mJ must separate, since the only other term it can combine with, F_4J, is independent of m. If separation is to occur for as few as two dis-

tinct values of m, it already implies that F_4J and mJ must separate into the sum of a function of u only and a function of v only. In this connection it should be noted that if we try to pick a new J for each m, the ratio of the J values will have to be a function of u only and also a function of v only, in view of the condition (see below) on the coefficients of X'' and Y''. Thus the ratio J_1/J could depend on m alone, and the above considerations apply.

5.1. The differentiated terms. It has been seen that our multiplier $J(u, v)$ is of the form $\alpha(u) + \beta(v)$. When (5.2) is multiplied by this, separation must occur, and hence the coefficients of X''/X and X'/X must depend on u alone, while those of Y''/Y and Y'/Y depend on v alone. The condition for the second derivatives tells us that

$$ds^2 = A^2\{[\alpha(u) + \beta(v)]\,[du^2 + dv^2] + dw^2\}$$

after a change of scale in the u and v co-ordinates. The condition on coefficients of the first derivatives allows us to assume that $R^2A = 1$, as we see by following the derivation of (1.7). Also we know from the first separation that A has the form (2.3), so that we obtain finally

$$ds^2 = \frac{[\alpha(u) + \beta(v)](du^2 + dv^2) + dw^2}{[t(u, v) + H(w)]^2}\,. \tag{5.3}$$

From the derivation it is clear that (5.3) is sufficient as well as necessary for separation.

In connection with (5.3) we note that du^2 and dv^2 have the same coefficients, just as in (1.5). The former result was obtained merely by introducing a change of parameters; it was not a consequence of separability. In the present case, on the contrary, a change of parameters is not permissible. The most we can do is make a change of scale, such as replacing u by a function of u. That the linear element has the same coefficients for du^2 and dv^2 is a result which had to be proved from separability of the equation. Also since the u and v co-ordinates cannot be changed at will, the methods formerly used to simplify the equations are not available here; but there is some compensation in that the form of F is now known.

5.2. The term free of unknowns. The foregoing results follow from consideration of the coefficients of the differentiated terms. From the other terms we obtain an equation analogous to (1.8), namely

$$(\alpha + \beta)F_4 = \alpha_1(u) + \beta_1(v). \tag{5.4}$$

In combination with (1.8), which we have seen is always satisfied, (5.4) gives

$$(\alpha + \beta)A^2(F_2 + G_2 + H_2) = (\alpha + \beta)h_4(w) + \alpha_1(u) + \beta_1(v). \tag{5.5}$$

Using (4.3) in place of $A^2(F_2 + G_2 + H_2)$ and taking $F^2 = \alpha + \beta$, we find from (5.5) that

$$-\frac{t_{uu} + t_{vv}}{8(t + H)} + \frac{H''(\alpha + \beta)}{4(t + H)} = -(\alpha + \beta)h_4 - \alpha_1 - \beta_1\,. \tag{5.6}$$

It may be shown that this relation, like (1.8), is a consequence of the others; we omit the details. The proof is closely analogous to that presented in full in Sec. 4.0.

5.3. Transformations leaving equations invariant. It is convenient to note the changes in functions or variables which leave the equations essentially unaltered. As before, use of such properties permits simplification of the methods used. By inspection

of the linear element (5.3), or of the equations themselves, we see that the following transformations will not lead to any essential change, if the k_i are constant:

$$t \to k_0 t, \tag{i}$$

$$\alpha \to k_1\alpha, \qquad \beta \to k_1\beta, \tag{ii}$$

$$\alpha \to \alpha + k_2\,, \qquad \beta \to \beta - k_2\,, \tag{iii}$$

$$u \to f(u), \qquad v \to g(v), \tag{iv}$$

$$\alpha \rightleftarrows \beta, \qquad u \rightleftarrows v, \tag{v}$$

DETERMINATION OF α AND β

6.0. An ordinary differential equation when $H' \neq 0$. Starting from (2.9), following through the derivation of (2.6) in detail, and using $F = (\alpha + \beta)^{1/2}$, we find that

$$c_1 = \frac{\alpha'' + \beta''}{2(\alpha + \beta)^2} - \frac{\alpha'^2 + \beta'^2}{2(\alpha + \beta)^3}\,, \tag{6.1}$$

where c_1 is constant. This result assumes $H' \neq 0$. In (6.1) we hold v constant and write α for $\alpha + \beta$ to obtain

$$2c_1\alpha^3 = \alpha(\alpha'' + c_2) - (\alpha'^2 + c_3),$$

which becomes

$$2c_1\alpha^3 = \alpha\left(p\,\frac{dp}{d\alpha} + c_2\right) - (p^2 + c_3)$$

when we take α instead of u as the independent variable, $p = \alpha'$ instead of α as unknown, and use $\alpha'' = p\,dp/d\alpha$. Upon introduction of a new variable $S = p^2 = \alpha'^2$, this in turn gives

$$\frac{dS}{d\alpha} - \frac{2S}{\alpha} - 4c_1\alpha^2 + 2c_2 - \frac{2c_3}{\alpha} = 0. \tag{6.2}$$

The integrating factor is $1/\alpha^2$ and leads to

$$S = \alpha'^2 = 4c_1\alpha^3 + 2c_2\alpha - c_3\,. \tag{6.3}$$

The symmetry of the equations noted in item (v) of Sec. 5.3 shows that we have a relation of the same type for β. Equation (6.1) becomes a polynomial in α and β, as we see by computing α'' and β'' from (6.3). Multiplying (6.1) by $(\alpha + \beta)^3$ we find that $\alpha'^2 + \beta'^2$ must be zero when $\alpha = -\beta$. It follows, then, that the equation corresponding to (6.3) for β is

$$\beta'^2 = 4c_1\beta^3 + 2c_2\beta + c_3\,. \tag{6.4}$$

Substitution into (6.1) leads to an identity, and hence there is no additional condition. Because the differential equation (6.3) also arises in the case $H' = 0$, as we shall see below, discussion of its solutions has been deferred to Sec. 6.4.

6.1. A partial differential equation when $H' = 0$. Turning now to the case in which H is constant, we absorb it into the function t, to obtain a simplified form of Eqs. (2.10)

and (2.9), Eqs. (2.4) and (2.11) remaining the same. If we take new independent variables $\phi = \log t$ and $\theta = \log F^2 = \log(\alpha + \beta)$ these relations assume a form involving derivatives only, not the unknown functions. Specifically from (2.4), (2.10), (2.11), (2.9) in that order we find the following:

$$2\phi_{uv} + 2\phi_u\phi_v = \theta_u\phi_v + \phi_u\theta_v , \tag{6.5}$$

$$\phi_u^2 + \phi_v^2 = \phi_{uu} + \phi_{vv} , \tag{6.6}$$

$$\phi_{uu} - \phi_{vv} + \phi_u^2 - \phi_v^2 = \theta_u\phi_u - \theta_v\phi_v , \tag{6.7}$$

$$\theta_{uu} + \theta_{vv} = 2(\phi_{uu} + \phi_{vv}). \tag{6.8}$$

Because of the form of θ we also have that

$$\theta_{uv} + \theta_u\theta_v = 0. \tag{6.9}$$

Let us differentiate the relation

$$2(\phi_u^2 + \phi_v^2) = \theta_{uu} + \theta_{vv} , \tag{6.10}$$

obtained from (6.6) and (6.8), with respect to u to find that

$$4(\phi_u\phi_{uu} + \phi_v\phi_{uv}) = \theta_{uuu} + \theta_{uvv} . \tag{6.11}$$

Upon eliminating the second derivatives ϕ_{uu} and ϕ_{uv} by use of the other relations, one finds that the terms involving ϕ all cancel, leaving the equation in $\theta[= \log(\alpha + \beta)]$

$$\theta_u(\theta_{uu} + \theta_{vv}) = \theta_{uuu} + \theta_{uvv} . \tag{6.12}$$

It will be seen that α and β can be determined from this.

6.2. Complete solution of the case $H' = \beta' = 0$. If β does not depend on v we may replace α by $\alpha + \beta$, as suggested in Sec. 5.3 item (iii), and β by $\beta - \beta = 0$. We have then

$$\theta_u\theta_{uu} = \theta_{uuu} \tag{6.13}$$

from (6.12), with θ equal to $\log \alpha$. Letting $\psi = \theta_u = \alpha'/\alpha$, substituting and integrating we get

$$\psi^2/2 = \psi' - c^2/2$$

with c constant. If $c \neq 0$ we have $\psi = c \tan(u/2 + c')$, but if $c = 0$ then $\psi = -2/(u + c')$. As the general solutions we thus find that

$$\alpha = \frac{c''}{\cos^2(uc + c')} \qquad \text{or} \qquad \alpha = \frac{c'}{(u + c)^2} . \tag{6.14}$$

In view of the equivalence (iv) of Sec. 5.3 the only essentially distinct cases are $\alpha = csch^2 u$, $csc^2 u$, or $1/u^2$.

6.3. An ordinary differential equation for the case $H' = 0$ but $\alpha'\beta' \neq 0$. We shall suppose now that $\alpha'\beta' \neq 0$, but retain the assumption that $H' = 0$. With θ replaced by its value $\log(\alpha + \beta)$, Eq. (6.12) now gives

$$\alpha'''(\alpha + \beta) - 4\alpha''\alpha' - 2\alpha'\beta'' + 3\alpha' \frac{\alpha'^2 + \beta'^2}{\alpha + \beta} = 0. \tag{6.15}$$

Proceeding as in the derivation of (6.2) we let u be constant to find that

$$\frac{dS}{d\beta} - \frac{3S}{\beta} + c_1\beta + c_2 + \frac{c_3}{\beta} = 0, \tag{6.16}$$

where $s = \beta'^2$ and β has been written for $\alpha + \beta$. The integrating factor is $1/\beta^3$ and it gives, after integration and multiplication by β^3,

$$S = \beta'^2 = A + B\beta + C\beta^2 + D\beta^3. \tag{6.17}$$

By the symmetry noted in (v), Sec. 5.3, we have also that

$$\alpha'^2 = a + b\alpha + c\alpha^2 + d\alpha^3, \tag{6.18}$$

where in both cases the coefficients are constant.

Inspection of (6.15) tells us that $\alpha'^2 + \beta'^2$ must be zero whenever $\alpha = -\beta$, since α''' and β'' are finite by (6.17) and (6.18). If α'^2 is given by (6.18), then β'^2 is uniquely determined as

$$\beta'^2 = -a + b\beta - c\beta^2 + d\beta^3. \tag{6.19}$$

It may be verified that (6.15) is now satisfied, and hence that there is no additional condition. Since these equations are equivalent to (6.3) and (6.4), obtained for the case $H' = 0$ (cf. Sec. 6.4), their solution completes the determination of α and β. It is seen, incidentally, that if $\alpha(u)$ satisfies (6.18), then the function $-\alpha(iv)$ will satisfy (6.19).

6.4. Canonical forms of the equation. Let us make use of Sec. 5.3 to simplify (6.18) and (6.19). Writing $-\alpha$ for α will make $d > 0$, if originally $d < 0$; and replacing α by $A\alpha + B$, u by u/C, we get

$$A^2\alpha'^2C^2 = a + b(A\alpha + B) + c(A\alpha + B)^2 + d(A\alpha + B)^3. \tag{6.20}$$

Supposing that d has been made greater than zero, we set

$$a + bB + cB^2 + dB^3 = 0, \tag{6.21}$$

$$b + 2cB + 3dB^2 = dA^2, \tag{6.22}$$

and choose $C = (Ad)^{1/2}$ to obtain the canonical form

$$\alpha'^2 = \alpha^3 + \lambda\alpha^2 + \alpha. \tag{6.23}$$

The corresponding substitution for β must be $\beta \rightarrow A\beta - B$ to be permissible; it gives (6.23) with α replaced by β and λ by $-\lambda$.

We see that A, B, and C may be assumed real (this is important), and that $A = 0$ is necessary only when there is a triple root of the original equation. These observations follow from the facts that the left-hand side of (6.22) is the derivative of (6.21) and that $d > 0$. For a triple root the canonical form is

$$\alpha'^2 = \alpha^3, \qquad \beta'^2 = \beta^3. \tag{6.24}$$

If we suppose now that $d = 0$ but $c \neq 0$ we find, in a similar way, that

$$\alpha'^2 = \alpha^2 + \lambda\alpha + 1, \qquad \beta'^2 = -\beta^2 + \lambda\beta + 1. \tag{6.25}$$

This form includes the double root case, which can be shown, however, to be impossible anyway. One would at first expect a $\pm$ sign in front of α'^2, but this is accounted for by interchanging α and β.

Next $c = d = 0$ gives

$$\alpha'^2 = \alpha, \qquad \beta'^2 = \beta \tag{6.26}$$

and $\alpha' = \beta' = 0$ is the last possibility. In Eq. (6.23) we must distinguish the three cases $|\lambda| < 2$, $|\lambda| = 2$, $|\lambda| > 2$, whereas in (6.25) we have only the one case $|\lambda| > 2$, since α' and β' must be real simultaneously.

6.5. Solution of the ordinary differential equations for α and β. We have seen that whenever H' or $\alpha'\beta' \neq 0$, the functions α and β must be solutions of one or another of the canonical differential equations in Sec. 6.4. These equations, though non-linear, may all be solved by elementary methods. The work is rather tedious, particularly since one must be careful to keep all possible solutions. We, therefore, content ourselves with a single example. From (6.23) we find that

$$du = [\alpha(\alpha^2 + \lambda\alpha + 1)]^{-1/2}\, d\alpha$$

which leads (among other possible expressions) to [14]

$$u + c = \left(\pm \frac{4}{r}\right)^{1/2} F\left[\frac{1}{r}(r - r')^{1/2}, \phi\right], \qquad \sin^2\phi = \frac{\alpha - r}{r' - r},$$

when the roots r, r', 0 are real and unequal, that is, when $|\lambda| > 2$. Here $F(x, p)$ is the elliptic function of the first kind. Since we want α as a function of u rather than the converse, we introduce the Jacobi elliptic functions to find that

$$\frac{\alpha - r}{r' - r} = \mathrm{sn}^2\left(\tfrac{1}{2}r^{1/2}(u + c), \qquad [(r - r')/r]^{1/2}\right)$$

with a similar expression for β. For r and r' we substitute the proper values in terms of λ to obtain, with a new u, a one-parameter family. This is for the case $0 < r' < \alpha < r$. Other cases are similarly treated, and lead to two additional expressions when $|\lambda| > 2$. The cases $|\lambda| < 2$, which gives conjugate complex roots, and $\lambda = 2$, which gives equal roots, are dealt with in the same way; the latter and all others in (6.24)—(6.26) give elementary functions.

DETERMINATION OF t

7.0. The partial differential equations. Up to this point we have shown that the linear element of orthogonal co-ordinates in Euclidean space must be of the form (5.3) if Laplace's equation is to separate. In addition H is one of the values (2.7) and α and β are each one of the values obtained in Secs. 6.0—6.5. It remains only to determine t in (5.3), and we proceed to this question forthwith.

Treating first the case $H' = 0$, our aim is to compute the expression involving F on the right of (2.9). Assuming that $\alpha'\beta'$ or $H' \neq 0$ we have (6.18) and (6.19). Differentiating both sides with respect to u and dividing by $2\alpha'$ we get

$$\alpha'' = \frac{3}{2}a\alpha^2 + b\alpha + \frac{c}{2} \tag{7.1}$$

with a similar expression for β''. Since $F = (\alpha + \beta)^{1/2}$ we have, for the expression desired,

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = \frac{1}{2}\left[\frac{\alpha'' + \beta''}{\alpha + \beta} - \frac{\alpha'^2 + \beta'^2}{(\alpha + \beta)^2}\right]$$

which reduces to

$$\left(\frac{F_u}{F}\right)_u + \left(\frac{F_v}{F}\right)_v = (\alpha + \beta)\frac{a}{4} \tag{7.2}$$

in view of (6.18), (6.19) and (7.1). Here a is the coefficient of α^3 in (6.18), and hence $a = 1$ for the cases with canonical equations (6.23), (6.24) but $a = 0$ for (6.25), (6.26).

The above assumes (6.18), which is valid only when H' or $\alpha'\beta' \neq 0$. Assuming now that $H' = \beta' = 0$ and writing α for $\alpha + \beta$ we find that the expression on the left of (7.2) becomes $(\alpha'/\alpha)_u/2$. For the case $H' = \beta' = 0$ it has been shown that α must be one of the three functions mentioned at the end of Sec. 6.2. By direct calculation we find that $(\alpha'/\alpha)_u/2 = \alpha$ every time, and hence (7.2) is valid with $a = 4$. Thus the expression is known in all cases.

When $H = 0$ we find that

$$t_{uu} + t_{vv} = \frac{t_u^2 + t_v^2}{t} + \frac{ta}{4}(\alpha + \beta), \tag{7.3}$$

$$t_{uu} + t_{vv} = 2\,\frac{t_u^2 + t_v^2}{t} \tag{7.4}$$

from (2.9) and (2.10), after making use of (7.2) and replacing F^2 by $\alpha + \beta$. These relations tell us that $t_u^2 + t_v^2 = 0$ whenever $a = 0$, so that in every non-cubic case if H is constant t must be constant also. The only remaining relations are (2.4) and (2.11), which become

$$2t_{uv} = \frac{\beta' t_u + \alpha' t_v}{\alpha + \beta}, \tag{7.5}$$

$$t_{uu} - t_{vv} = \frac{\alpha' t_u - \beta' t_v}{\alpha + \beta}, \tag{7.6}$$

for the present situation $F^2 = \alpha + \beta$. Equations (7.5) and (7.6) are valid for all H.

When H is not zero the corresponding form of (2.9) and of (2.10) is obtained by simply setting $F^2 = \alpha + \beta$ in the relations of Secs. 3.1—3.6. Specifically, if $H = w^2$ we have that

$$t_{uu} + t_{vv} = 4(\alpha + \beta), \tag{7.7}$$

$$t(t_{uu} + t_{vv}) = t_u^2 + t_v^2\,; \tag{7.8}$$

if $H = e^w$ then $t = 0$; if $H = \cosh w$, then

$$t_{uu} + t_{vv} + 2t(\alpha + \beta) = 0, \tag{7.9}$$

$$t_u^2 + t_v^2 + (\alpha + \beta)(t^2 - 1) = 0; \tag{7.10}$$

and if $H = \sin w$, then

$$t_u^2 + t_v^2 = (\alpha + \beta)(t^2 - 1), \tag{7.11}$$

$$t_{uu} + t_{vv} = 2(\alpha + \beta)t. \tag{7.12}$$

It may be verified that both (2.9) and (2.10) are satisfied identically when, for a given H, the function t satisfies the appropriate pair of equations from (7.3), (7.4) or (7.7)—(7.12). These relations plus (7.5) and (7.6), then, give the necessary and sufficient condition for separation in Euclidean space.

7.1. An indirect method of solving the equations for t. To obtain all possible linear elements we must have all possible values of t; in other words we must have the general solution of the above differential equations. This general solution is not easy to obtain directly and hence we use an indirect procedure, which depends on the fact that α and β are already known for every case.

Given a particular linear element ds^2 we note from (5.3) that all others using the same α and β are obtained from this one by means of a conformal transformation, with ds^2 now regarded as the metric of Riemannian V_3. Thus, since α and β are to be the same in both cases we can change t and H only. The two elements are, consequently, proportional. By a theorem of Liouville [15], the most general conformal transformation which preserves the Euclidean character of the space—i.e., preserves the relations of Sec. 2—is a rigid motion, a reflection, or an inversion. Only the latter is of interest here. Hence given α and β, if we can somehow discover just one admissible t and H, then all others can be found from this by inversion.[4]

A single value of t is found without much trouble from the equations; in many cases it is obtained by inspection. Or one may refer to [3] and [11], which give examples of Euclidean linear elements with our α and β for each case. Knowing the single solution, we get the others by inversion as outlined above, and it remains only to see whether all inversions are permissible.

CONCLUDING REMARKS

8.0. Concerning inversion. It was found by actual trial that all inversions of permissible co-ordinates were themselves permissible, if by *permissible* we mean that the space is Euclidean and the equation separates. The new linear element had the form (5.3) and the other conditions (7.3)-(7.12) were always satisfied, independently of the center of inversion. We propose now to give a direct proof that this must necessarily be the case.

First, since the original linear element was permissible, it had the form (1.5) with $F^2 = \alpha + \beta$. Hence, because an inversion is a conformal transformation, the new linear element will also have this form with a different A, say $\bar{A}$. The equation will separate if $\bar{R}$, the new value of R, is given by $\bar{R}^2\bar{A} = 1$. This is true provided only that the relations (1.8) and (5.5) continue to hold in the new system, and we shall see that such is the case.

Since the original linear element was permissible, it was obtained from a co-ordinate system in Euclidean space. Consequently the new one, having been found by inversion, will have the same property. Now the relations (1.8) and (5.5), which are the only remaining conditions for separation of the equation, have been proved to be satisfied automatically whenever the space is Euclidean. They are therefore satisfied by the new linear element, and hence the equation separates. Moreover the fact that A has the form (2.3) was also deduced from the Euclidean relations, and hence will persist for $\bar{A}$; and the same is true of all the differential equations for t and H. Thus the linear

[4]A similar use of Liouville's theorem is made in [8].

element obtained by inversion satisfies every one of our conditions if the original linear element does. These remarks, which apply to partial as well as complete separation, complete the proof of the fourth result:

If the above procedure can be applied to the resulting differential equation for S to get further separation, so that the solutions have the form $R(u, v, w)X(u)Y(v)Z(w)$, then the co-ordinates must all be orthogonal and must be either toroidal co-ordinates, or the well-known cases (I—III *but not* IV *in* [3]) *giving separation of this type with $R = 1$, or else the co-ordinates obtained by inversion of these in a sphere.*

We have also proved that:

The set of co-ordinates giving separation RSZ or $RXYZ$, as the case may be, is closed under inversion.

In other words, if a given co-ordinate system is in the set, then the system obtained by inverting it in any sphere will likewise be in the set. This behavior of course is not found when we confine ourselves to the case $R = 1$.

It must be mentioned that these results on inversion, though interesting mathematically, are of slight practical importance. The way one would actually solve Laplace's equation in co-ordinates which are inversions of standard cases would be to solve the standard case and then invert the solution. The fact that the equation could be solved directly by separation of variables in the new co-ordinate system, though true, would not be used in practice.

8.1. Conditions for which $R = 1$. We shall have $R = 1$ when, and only when, it can be absorbed in the product XYZ, that is, when $R = p(u)q(v)r(w)$. In view of the condition $R^2A = 1$ in Eq. (1.7) we see that A too must have this form. Such a condition, when combined with (2.3), makes H constant and F of the form $r(u)s(v)$, or else it makes t constant. It is found that these conditions hold only in cases which occur in [3]. Thus,

Laplace's equation separates in the sense XYZ when, and only when, the wave equation so separates.

The "when" part of this result is well-known, but the "only when" part is believed to be new.

8.2. The wave equation. In Sec. 4.3 we considered the conditions under which the wave equation would separate partially, to give solutions of the type RSZ. Turning now to the case in which there is complete separation $RXYZ$, we encounter a difficulty when we try to show that $F^2 = \alpha(u) + \beta(v)$. The previous argument depended on the fact that one term of (5.2) involved the separation constant m while the other did not. For the wave equation, however, we may have k depending on m, and this method therefore cannot be used. Confining our attention to the case $H = 0$, which we have seen is the only case in which F need not be constant in the wave equation, we shall re-examine the separation procedure.[5]

Since Sec. 5.1 makes $J = F^2 = G^2$, the same equations are found, in general, with F^2 replacing $\alpha + \beta$. But (5.4) becomes

$$F^2\{A^2(F_2 + G_2 + H_2 + k) - m\} = \alpha_2(u) + \beta_2(v) \tag{8.1}$$

when F_4 is replaced by the F_5 of (4.12). Observe that $H = 0$ in (2.3) makes (1.3) depend

[5]This discussion was revised in proof (April, 1949); the earlier version was incorrect unless we had $k_1m_2 = k_2m_1$ in (8.2).

on u and v alone, by (1.5) and (1.7). Hence H_3 of (4.12) is constant, and may be absorbed in F_3.

If there is separation for three pairs of values (m_i, k_i) such that

$$\begin{vmatrix} 1 & m_1 & k_1 \\ 1 & m_2 & k_2 \\ 1 & m_3 & k_3 \end{vmatrix} \neq 0, \tag{8.2}$$

then the three equations (8.1) may be solved for F^2, whence we obtain the desired result $F^2 = \alpha + \beta$.

When the determinant (8.2) is zero, which can happen in practice (cf. Ref. [16]), the complete result has not been obtained. It appears that separation can occur for co-ordinates not obviously reducible for the known cases, and that $J = \alpha + \beta$ is no longer essential. Also a rather long investigation shows that $\theta = \log F^2$ must satisfy $\theta_{uu} + \theta_{vv} = cF^2$ when $F^2 \neq \alpha + \beta$, and that the constant c is zero if any minor like $m_1k_2 - m_2k_1$ vanishes. This latter condition gives $t = 0$ by (6.6) and (6.8), whence we conclude that $F^2 = \alpha + \beta$. The general problem in which the determinant (8.2) vanishes, but no minor does, is reserved for later investigation.

Now that we have obtained the proper form for F we can use all the results of the preceding sections. These results, which certainly represent a necessary condition on the co-ordinates for separation of the wave equation, tell us that we have t or H constant only in the well-known cases giving $R = 1$. Consequently,

The wave equation separates in the sense $RXYZ$ only in the known cases for which it separates in the sense XYZ, provided (8.2) *holds.*

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LINEAR EQUATION SOLVERS*

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1. There are two types of devices for the solution of simultaneous linear equations which have been developed. Suppose the given system of equations is

$$\sum_{j=1}^{n} a_{ij}x_j = b_i , \qquad i = 1, 2, \cdots , n. \tag{1}$$

In one type the b_i are fed into the machine in such a way as to drive the unknowns x_j to their correct values. In the second type, the x_j are not driven by power supplied from the constant inputs but reach an equilibrium situation corresponding to the solution by a process of adjustment. We are concerned in this article with the operating conditions for this last type of machine when the adjusting process is determined by a linear operator with constant coefficients.

We suppose that each a_{ij} can run independently through a real range which contains the origin. Thus, the determinant may be zero and any matrix can be represented by a suitable choice of scale for the unknowns and the b_i . Adjusting machines which are stable even when the determinant is zero may be designed. For instance, a block diagram is given in the author's book[1] for such a machine. Another example is the set-up described by Goldberg and Brown[2] which will insure stability when a certain type of feedback is used.

However, in each case the coefficient network is duplicated. In the present article, we point out that if an adjusting type of machine is to operate successfully whenever the determinant A is not zero, then the square of the determinant must enter the indicial equation of the equations of motion for the machine. This necessary condition for successful operation rules out any linear feedback which does not involve using the a_{ij} twice. This result generalizes certain aspects of the necessity argument indicated in Goldberg and Brown.

In Secs. 2 and 3 below, we describe precisely the type of machine we are concerned with. These machines may function continuously or in discrete steps. In Secs. 4 and 5 we obtain necessary and sufficient conditions that the machine should operate successfully in all cases where a solution is uniquely determined. These conditions are analogous to stability conditions for a linear network. In the case of a continuous machine, this analogy is readily established; in the case of a discrete step machine, these operational conditions are obtained by considering certain parts of the theory of linear difference equations. In Sec. 6 we prove the mathematical theorem upon which our result is based. It is shown in Sec. 7 that an adjusting machine with a linear feedback network, which is independent of the coefficients of the equations, will not always operate successfully. Section 8 contains the mathematical basis for Sec. 5 which is concerned with discrete step machines.

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[1] F. J. Murray, *Mathematical machines*, Kings Crown Press, New York, 1947 p. 92 (1st ed.), pp. III 20-21 (2nd ed.).

[2] E. A. Goldberg and G. W. Brown, J. Appl. Phys. **19**, 339-344 (1948).

2. A mathematical machine can be regarded as a combination of computing components, each of which performs a specific mathematical operation. Each component has various inputs and a specific output. A combination of components can be used to evaluate a formula or function of the inputs. In a machine for solving a system of linear equations

$$\sum_i a_{ij}x_j + b_i = 0 \tag{1}$$

the coefficients a_{ij} and the constants b_i are inputs whose values do not vary during the operation of the machine.

In an adjusting type machine for solving linear equations, there are variables to represent the x_i. Each of these is associated with a unit of the following character. Each unit has an input X_i and an output x_i. The output x_i is suitable for use as an input in the computing components of the machine and the input X_i must correspond to an output of computing components. The relation between the input X_i and the output x_i is an operational one, $L_i(x_i) = X_i$. Various possibilities for L_i are discussed below; L_i may depend on i, it will always be linear.

The adjusting type device functions as follows. While the x_i units are inactivated, the a_{ij} and the b_i are entered into the machine. Presumably at this point the values of the x_i do not constitute a solution of the given system of equations (1). Now, however, the x_i units are activated. Various combinations of components compute the errors in each equation

$$\epsilon_i = \sum_{j=1}^{n} a_{ij}x_j + b_i \tag{2}$$

and these, in turn, are used to compute the X_i as some function $f(\epsilon_1, \cdots, \epsilon_n)$ of the errors. The inputs X_i of the x_i units cause the latter to vary so as to approach the solution of the system (1). Since as we have pointed out in Sec. 1, for this purpose certain f values exist which are linear combinations of the ϵ_i, we will assume that f is linear in the ϵ_i:

$$X_j = \sum_i k_{ij}\epsilon_i \tag{3}$$

where the k_{ij} are constants in the sense that they do not change during the process in which the x_i are adjusted to the correct value.

3. We have not specified the nature of the operators L_i,

$$L_i(x_i) = X_i, \tag{4}$$

or the method of functioning for the components. In general, there are two ways or manners in which a machine of this type may operate: (A) The adjusting process proceeds continuously. Each component has continuous inputs and output and the L_i are differential operators. We suppose the coefficients in each L_i are constants and that the coefficient of the highest derivative is 1. (B) The adjusting process proceeds in discrete steps. L_i is a linear difference operator with constant coefficients but the components may have discrete inputs or continuous inputs. Again the coefficient of the highest order difference in L_i is 1.

The two examples cited in the introduction operate in the (A) manner. An electronic digital computer programmed to operate in the sequence indicated by Eqs. (2), (3),

(4) would operate like (B). A non-essential generalization permits us to include the manually adjusted type of equation solver such as those described in the author's book *Mathematical machines* (2nd edition, pp. III 16-20; 1st ed., pp. 87-91). A full cycle of the adjusting process in these machines corresponds to one step in the sense of (B) above. However, the adjusting equations (4) are to be replaced by a set of relations in the form

$$\begin{aligned} L_1(x_1) &= X_1 , \\ L_2(x_1 , x_2) &= X_2 , \\ &\cdots \\ L_n(x_1 , \cdots , x_n) &= X_n , \end{aligned} \tag{4'}$$

where each L is linear in the x_i . It will be seen that the generalization represented by (4′) does not affect our argument. One may mention that L_1 may be taken simply as the differencing operation Δ. In general L_i contains x_i only in the form Δx_i so that the first equation determines Δx_1 , the second Δx_2 and so forth.

We are interested then in the type of machine represented by the sequence of equations (2), (3), (4) or (4′). The obvious problem that appears here is concerned with the choice of the k_{ij} in (3) and the L_i in (4) so that the machine will work, i.e. so that the machine will adjust itself to a solution. As we have pointed out, sufficient conditions for adequate operation in all circumstances are known, but these require that the k_{ij} depend on the a_{ij} . This means that the a_{ij} must be used in computing ϵ_i and also again in forming the X_i .

Our objective is to establish that this double use of the a_{ij} is necessary if the machine is to function in all cases in which there is a solution. We suppose that the a_{ij} are permitted to assume independently all values in an interval which includes zero, say, for instance, from minus 1 to plus 1. If this is true then, by suitable choice of scale for the equations and the unknowns, any system of equations can be represented in the machine.

The above statement seems to neglect the case in which the a_{ij} appear digitally or as decimal fractions. However, when we are given a device in which the a_{ij} vary discretely in the components, we can regard each component as replaced by a continuous component of perfect accuracy and apply our argument to it. Now suppose in the idealized machine, we find a region of non-operation or instability. Then, in general, the original discrete machine will have permissible values for the a_{ij} which fall in this region and, of course, it will also be unstable.

Let us end this section by pointing out certain conclusions concerning the above mathematical setup which one may reach from the assumption that the machine will operate successfully as an adjusting device.

1) If the machine operates in the manner (A) and the L_i are differential operators such that

$$L_i(x_i) = x_i^{(t)} + l_{1i}x_i^{(t-1)} + \cdots + l_{ti}x_i ,$$

then $l_{ti} = 0$.

2) If the machine operates in manner (B) and

$$L_i(x_i) = \Delta^t x_i + l_{1i}\Delta^{t-1}x_i + \cdots + l_{ti}x_i ,$$

then $l_{ti} = 0$.

From Eqs. (3) and (4), in either case, we get

$$L_i(x_i) = \sum_j k_{ij}\epsilon_j . \tag{5}$$

Now suppose we are solving a system of equations in which x_i is not zero. Then after an adequate time interval in a successfully operating machine all the ϵ_i will be small, x_i will be close to its true value, all the differencing operators Δ^k and differentiating operators $(d/dt)^k$ will yield a small result, and these equations will be inconsistent with the assumption that $l_{ti} \neq 0$. The case (4′) is readily treated on an inductive basis.

4. Consider the Eqs. (5) in the case (A):

$$\begin{aligned} L_i(x_i) &= x_i^{(t)} + \cdots + l_{t-1,i}x_i^1 \\ &= \sum_{j=1}^n k_{ij}\epsilon_j \\ &= \sum_{j=1}^n k_{ij}\Big(\sum_{\alpha=1}^n a_{j\alpha}x_\alpha + b_j\Big). \end{aligned} \tag{5A}$$

Equation (5A) can be written in the form

$$L_i(x_i) - \sum_\alpha \Big(\sum_j k_{ij}a_{j\alpha}\Big)x_\alpha = \sum_j k_{ij}b_j = \text{(say) } B_i .$$

The operators L_i can be treated as numbers, and we may employ the process by which Cramer's Rule is usually established to eliminate all but one unknown x. Consequently each x satisfies a differential equation in the form

$$\nabla x_i = C_i \tag{6}$$

where ∇ is a differential operator,

$$\nabla x_i = x_i^{(m)} + D_1x_i^{(m-1)} + \cdots + D_mx_i ,$$

and each C_i is a constant which of course depends on the b_i. The coefficients D do not depend on i. Clearly D_m is the determinant of the matrix with elements $\sum_j k_{ij}a_{j\alpha}$. (This follows from the fact established in the previous section that l_{ti} is zero.) Thus

$$D_m = KA, \tag{7}$$

where K and A are, respectively, the determinants of the matrices with elements k_{ij} and a_{ij}.

We proceed next to obtain the condition mentioned in Sec. 1 for successful operation in case (A). This condition is concerned with the algebraic equation

$$\mu^m + D_1\mu^{m-1} + \cdots + D_m = 0, \tag{8}$$

which is usually referred to as the indicial equation of the homogeneous differential equation $\nabla x_i = 0$.

We first obtain the general solution of $\nabla x_i = 0$. Suppose $\mu_1, \cdots, \mu_{s_1}$ are the real roots of (8) and that μ_j has multiplicity r_j. Also suppose that $\alpha_1 + i\beta_1, \alpha_1 - i\beta_1, \cdots, \alpha_{s_2} + i\beta_{s_2}, \alpha_{s_2} - i\beta_{s_2}$ are the complex roots of (8) and suppose that $\alpha_i + i\beta_i$ has multi-

plicity u_j. Then it is well-known that the general solution x_i^* of $\nabla x_i = 0$ can be written in the form

$$x_i^* = \sum_{k=1}^{s_1} \sum_{j=1}^{n_k} M_{j,k} t^{j-1} \exp(\mu_k t) + \sum_{k=1}^{s_2} \sum_{j=1}^{n_j} t^{j-1} \exp(\alpha_k t)(P_{j,k} \cos \beta_k t + Q_{j,k} \sin \beta_k t). \quad \text{(9A)}$$

On the other hand if s_0 is such that D_{m-s_0} is the D with highest subscript which does not vanish, then a particular solution of (6) is obtainable in the form $X_{s_i} t^{s_0}$ and the general solution of (6) can be written in the form

$$x_i = X_{s_i} t^{s_0} + x_i^*. \quad \text{(10)}$$

Now suppose that the system of equations has a unique solution in which no unknown Y_i is zero. For the machine to function correctly under these circumstances, each x_i must approach the correct constant value Y_i as t approaches infinity. Now one can show that an expression of the type (10) will approach a constant value as t approaches infinity if, and only if, $s_0 = 0$ and all the μ_k and α_k are negative, i.e. if (8) is stable.

Thus we have shown that:

An adjusting machine of type (A) *will operate successfully if, and only if, all the roots of the indicial equation* (8) *lie in that half of the complex plane for which the real part of a number is negative, whenever the system of equations is non-singular.*

5. We now wish to go through the analogous discussion for case (B). In this case, the L_i are difference operators

$$L_i(x_i) = \Delta^t x_i + l_{1,i} \Delta^{t-1} x_i + \cdots + l_{t-1,i} \Delta x_i ,$$

and we obtain

$$L_i(x_i) = \sum_{j=1}^{n} k_{ij} \epsilon_j . \quad \text{(5B)}$$

As before, we also obtain by elimination

$$\nabla x_i = C_i , \quad \text{(6B)}$$

where

$$\nabla x_i = \Delta^m x_i + D_1 \Delta^{m-1} x_i + \cdots + D_m x_i .$$

The statements concerning D_i and D_m are the same as before and, in particular, (7) holds.

The customary method of handling a homogeneous difference equation $\nabla x_i = 0$ with constant coefficients is to look for solutions in the form $x_i(t) = a^t$. For these we have $\Delta x_i = a^{t+1} - a^t = (a - 1)a^t = (a - 1)x_i(t) = \mu x_i(t)$ for $\mu = a - 1$. Thus if we permit μ to have this meaning, we can obtain the indicial equation

$$\mu^m + D_1 \mu^{m-1} + \cdots + D_m = 0. \quad \text{(8B)}$$

If we had two machines, one of which operated in manner (A) and the other in manner (B) with, however, the L_i and X_i being such that the coefficients L_{ij} and k_{ij} were the same in each case, then the Eq. (8) would also be the same for each case.

Corresponding to every distinct solution of (8B) we have a solution a^t, where $a = \mu + 1$. For multiple roots the same technique as that of ordinary differential equations

permits one to find r linearly independent solutions if the root has multiplicity r. For instance, if μ_0 has multiplicity 3 then there is a polynomial ϕ such that

$$\mu^m + D_1\mu^{m-1} + \cdots + D_m = (\mu - \mu_0)^3\phi(\mu)$$

and

$$\nabla(\mu + 1)^t = (\mu - \mu_0)^3\phi(\mu)(\mu + 1)^t.$$

From this we can infer that

$$\nabla(\mu_0 + 1)^t = 0$$

$$\frac{\partial}{\partial\mu}[\nabla(\mu + 1)^t]_{\mu=\mu_0} = 0,$$

$$\frac{\partial^2}{\partial\mu^2}[\nabla(\mu + 1)^t]_{\mu=\mu_0} = 0.$$

Since Δ and $\partial/\partial\mu$ are commutative operations, we can infer that

$$\nabla\left(\frac{\partial}{\partial\mu}[(\mu + 1)^t]_{\mu=\mu_0}\right) = 0$$

and

$$\nabla\left(\frac{\partial^2}{\partial\mu^2}[(\mu + 1)^t]_{\mu=\mu_0}\right) = 0.$$

Thus $(\mu_0 + 1)^t$, $t(\mu_0 + 1)^{t-1}$ and $t(t - 1)(\mu_0 + 1)^{t-2}$ are solutions of $\nabla x_i = 0$ when μ_0 is a triple root of the indicial equation. Since linear combinations of solutions are also solutions $(\mu_0 + 1)^t$, $t(\mu_0 + 1)^t$ and $t^2(\mu_0 + 1)^t$ are equivalent to the given set if $\mu_0 + 1$ does not equal zero. If $\mu_0 + 1 = 0$, then $x_i = 1$ is a solution; if $\mu_0 = -1$ is a root of multiplicity 3 of (8), then, of course, the analogous equation for a has zero as a triple root and thus $x_i = 1$, t and t^2 are solutions. If μ_0 is a complex root and the D_i are real then $\bar{\mu}_0$ is also a root and solutions are obtained by taking the real and imaginary parts of the above solutions.

It is, of course, characteristic of the theory of linear difference equations that it parallels the theory of linear differential equations except that constants are replaced by periodic functions in the solutions. (There is another departure indicated below.) However, we need only consider the variable t to have discrete values 0, 1, 2, $\cdots$; then periodic functions need not enter our discussion. For this situation we give a sequence of lemmas which yield results analogous to those we have used in the differential equation case. These are proved in Sec. 8 below.

Lemma A. *Let*

$$\nabla x = 0 \tag{6B'}$$

be a linear difference equation with constant coefficients. Let the variable t take on only the values 0, 1, 2, $\cdots$ *so that a solution of* (6B′) *is an infinite sequence*

$$\{x(0), x(1), x(2), \cdots\}.$$

There are m linearly independent real solutions of (6B′).

COROLLARY. *For the solutions obtained in Lemma A, we can suppose that the first determinant is not zero.*

LEMMA B. *Under the hypothesis of Lemma A, every solution of* (6*B'*) *is a linear combination of the m linearly independent solutions obtained in Lemma A.*

Lemma B specifies the solution of the homogeneous equation $\nabla x_i = 0$; we can readily show that this is in a form analogous to (9A). For to each solution μ of (8) we can find a principal value of the logarithm such that

$$1 + \mu = \exp(\alpha' + i\beta')$$

or if $1 + \mu$ is real and positive, we can write simply

$$1 + \mu = \exp(\mu').$$

If s_0 is again such that D_{m-s_0} is the D with highest subscript which is not zero, then a particular solution of (6B) is readily obtained in the form

$$x_i = X_{s_0 i} \frac{t!}{s_0!(t - s_0)!}.$$

The argument used in the previous section now shows that if the machine is to function satisfactorily, then when the solution is unique one must have all μ' and α' negative. However, we are now one step removed from Eq. (8) and the condition for satisfactory operation of the machine is now that all the roots of (8) should be in the unit circle with center at $\mu = -1$. Thus we have proved:

An adjusting machine which operates in the manner (B) *will operate successfully if, and only if, all the roots μ of the indicial equation* (8) *lie in the unit circle with center at $\mu = -1$, whenever the determinant of a system* (A) *is not zero.*

6. We have established two facts which hold for adjusting machines quite irrespective of whether they operate in manner (A) or (B).

a) *The last coefficient D_m of the indicial equation is in the form KA where A is the determinant of the given system of equations.*

b) *If any root μ of Eq.* (8) *has a positive real part when the given system of equations is non-singular, then the machine will not operate successfully for this system.*

We now wish to establish the following theorem upon which our conclusions are based.

THEOREM. *Let Eq.* (8) *be the indicial equation of the Eq.* (6) *obtained by eliminating all but one x_i from the Eqs.* (4). *Furthermore, let us suppose that if the system of equations* (1) *is non-singular, then none of the roots μ of the indicial equation* (8) *has positive real parts. Then D is divisible by A^2, where A is the determinant of the system of equations* (1).

PROOF. Let us suppose that our system of equations has been chosen in the following way. Let $A(\lambda)$ be the characteristic determinant of the matrix and suppose we choose a set of a_{ij} so that all the latent roots, i.e. the roots of $A(\lambda) = 0$, are real and distinct. A symmetric real matrix of this type is readily found. Furthermore, for real a_{ij} one can show that for small changes in the a_{ij} the roots remain real and distinct. This is readily seen from the graph of $A(\lambda)$ which will have $n - 1$ maxima and minima whose ordinates alternate in sign if, and only if, $A(\lambda) = 0$ has n distinct roots. Thus we have an open

region in the n^2 dimensional a_{ij} space in which the characteristic roots are real and distinct.

Let us, however, for the moment regard all the a_{ij} as fixed but consider all systems of equations in the form

$$(a_{11} - \lambda)x_1 + \cdots + a_{1n}x_n + b_1 = 0,$$
$$\cdots \tag{1.λ}$$
$$a_{n1}x_1 + \cdots + (a_{nn} - \lambda)x_n + b_n = 0.$$

Equation (8) can now be written in the form

$$\mu^m + D_1(\lambda)\mu^{m-1} + \cdots + D_m(\lambda) = 0,$$

where now the $D_k(\lambda)$ are polynomials in a_{ij} and λ. We know that $D_m(\lambda) = KA(\lambda)$. ($K$ is constant, but only in the sense that it does not change during the operation of the machine, i.e. in the sense a_{ij} and λ are constants.)

Now suppose λ' is a root of $A(\lambda) = 0$. Then since $D_m(\lambda) = KA(\lambda)$, $\mu = 0$ is a root of the indicial equation for the system of equations (1.λ). Let p be the last integer such that $D_{m-p}(\lambda')$ does not equal zero. Since D_{m-k} is a polynomial in λ, $D_{m-k}(\lambda) = (\lambda - \lambda')\overline{D}_{m-k}(\lambda)$ for $k < p$ where $\overline{D}_{m-k}(\lambda)$ is also a polynomial in λ.

Now suppose $\overline{D}_m(\lambda') \neq 0$. Let us make the usual construction of the Newton polygon in order to obtain a series expansion[3] of the roots μ of the indicial equation in powers of $\lambda - \lambda'$:

$$\mu = c_1(\lambda - \lambda')^e + c_2(\lambda - \lambda')^{2e} + \cdots .$$

We readily find that (0, 1) and (p, 0) are vertices of the polygon. Let $\theta_1, \cdots, \theta_p$ denote the p roots of the equation

$$\overline{D}_m(\lambda') + \theta^p D_{m-p}(\lambda') = 0.$$

We have for each $k = 1, \cdots, p$ a solution in the form

$$\mu_k = \theta_k(\lambda - \lambda')^{1/p} + \cdots .$$

Now if $p = 1$, θ_1 is real and $\theta_1(\lambda - \lambda')$ changes sign as λ varies through λ'. This would yield a μ with a positive real part when $A(\lambda)$ is not zero. But this violates our hypothesis. For $p = 2$, we can choose λ so that both $\theta_1(\lambda - \lambda')^{1/2}$ and $\theta_2(\lambda - \lambda')^{1/2}$ are real and one will be positive. This will again destroy the stability. For $p = 3, 4, \cdots$ we can take $\lambda - \lambda'$ positive and since at least one θ_k has a positive real part, the real part of μ will be positive, and again we have contradicted the hypothesis.

Thus the assumption that $\overline{D}_m(\lambda') \neq 0$ contradicts the hypothesis that the roots have negative real parts when A is not zero. Hence $\overline{D}_m(\lambda') = 0$ and D_m has a factor $(\lambda - \lambda')^2$.

Now let us divide $D_m(\lambda)$ by $A^2(\lambda)$, considering them as polynomials in λ and the a_{ij}. Since the coefficient of λ^{2n} in $A^2(\lambda)$ is one, we find that there are two polynomials Q and R such that

[3]Cf., for instance, K. W. S. Hensel and G. Landsberg, *Theorie der Algebraischen Funktionen Einer Variabeln*, Teubner, Leipzig, 1902, pp. 39-52.

$$D_m(\lambda) - Q(\lambda)A^2(\lambda) = R(\lambda),$$

where $R(\lambda)$ has degree less than $2n$ in λ.

This holds for all matrices $\{a_{ij}\}$ in the $\{a_{ij}\}$ region we decided upon, within which all the roots λ' are real and distinct. However, in this region for each choice of $\{a_{ij}\}$, $R(\lambda)$ is divisible by $(\lambda - \lambda')^2$ for every root λ' of $A(\lambda) = 0$, and since the degree of $R(\lambda)$ is less than $2n$, $R(\lambda)$ must be identically zero in λ for $\{a_{ij}\}$ fixed. Since this holds for every matrix $\{a_{ij}\}$ of the region, R must be zero as a polynomial in a_{ij} and λ. This establishes the theorem.

7. The above necessary criterion for successful operation permits us to show that a simple adjusting machine, with a linear feedback in which the a_{ij} are not used, cannot function successfully for all systems for which the determinant is not zero. For here $D_m = KA$ where K does not depend on the a_{ij} and thus D_m is not divisible by A^2.

8. Lemma A. *Let*

$$\nabla x = 0 \tag{6B$'$}$$

be a linear difference equation with constant coefficients. Let the variable t take on only the values 0, 1, 2, $\cdots$ *so that a solution of* (6B′) *is an infinite sequence*

$$\{x(0), x(1), x(2), \cdots\}.$$

There are m linearly independent real solutions of (6B′).

Let us first ignore the restriction that the solutions be real. The above argument indicates how m solutions can be obtained by means of the indicial equation. We must establish then the linear independence of the m solutions obtained:

$$\begin{array}{ll} \{x^{(1)}(0), & x^{(1)}(1), \cdots\}, \\ \{x^{(2)}(0), & x^{(2)}(1), \cdots\}, \\ \cdots & \\ \{x^{(m)}(0), & x^{(m)}(1), \cdots\}. \end{array}$$

Regarding these solutions as an infinite matrix, we must show that the rank is m. If we do this, a finite submatrix must have rank m and have linearly independent rows. Consequently, the full matrix has linearly independent rows.

If the roots of (8) are all distinct then, of course, the determinant of the first m columns is the well-known Vandermonde determinant

$$\begin{vmatrix} 1 & \mu_1 + 1 & \cdots & (\mu_1 + 1)^{m-1} \\ 1 & \mu_2 + 1 & \cdots & (\mu_2 + 1)^{m-1} \\ \cdots & & & \\ 1 & \mu_m + 1 & \cdots & (\mu_m + 1)^{m-1} \end{vmatrix} = \prod_{i>j} (\mu_i - \mu_j),$$

which is not zero.

The case of multiple roots is treated by induction. We suppose that μ_0 is a triple root and that the first three solutions correspond to μ_0. In addition, for the moment we suppose that all other roots are distinct. Then the first three rows of the matrix are

$$\begin{Bmatrix} 1 & \mu_0 + 1 & (\mu_0 + 1)^2 & (\mu_0 + 1)^3 & (\mu_0 + 1)^4 & \cdots \\ 0 & 1 & 2(\mu_0 + 1) & 3(\mu_0 + 1)^2 & 4(\mu_0 + 1)^3 & \cdots \\ 0 & 0 & 2\cdot 1 & 3\cdot 2(\mu_0 + 1) & 4\cdot 3(\mu_0 + 1)^2 & \cdots \end{Bmatrix}.$$

In the distinct root case, we have

$$\begin{vmatrix} 1 & \mu_0 + 1 & \cdots & (\mu_0 + 1)^{m-1} \\ 1 & \mu_1 + 1 & \cdots & (\mu_1 + 1)^{m-1} \\ 1 & \mu_2 + 1 & \cdots & (\mu_2 + 1)^{m-1} \\ \cdots & & & \end{vmatrix}$$

$$= (\mu_0 - \mu_1)(\mu_0 - \mu_2)(\mu_1 - \mu_2) \prod (\mu_0 - \mu_i) \prod (\mu_1 - \mu_i) \prod (\mu_2 - \mu_i) \prod (\mu_i - \mu_j)$$

where i, j do not equal 0, 1, 2. If we let $\mu_1 = \mu_0 + h$, the elements in the second row of the determinant become

$$(\mu_1 + 1)^j = (\mu_0 + 1)^j + h[j(\mu_0 + 1)^{j-1}] + 1/2\, h^2[j(j-1)(\mu_0 + 1)^{j-2}] + \cdots + h^j. \quad (11)$$

Now we can subtract the first row of the determinant from this second row, divide both sides by h and take the limit as h approaches zero. Then we obtain

$$\begin{vmatrix} 1 & \mu_0 + 1 & (\mu_0 + 1)^2 & \cdots & (\mu_0 + 1)^{m-1} \\ 0 & 1 & 2(\mu_0 + 1) & \cdots & (m-1)(\mu_0 + 1)^{m-2} \\ 1 & \mu_2 + 1 & (\mu_2 + 1)^2 & \cdots & (\mu_2 + 1)^{m-1} \\ \cdots & & & & \end{vmatrix}$$

$$= (\mu_0 - \mu_2)^2 \prod (\mu_0 - \mu_i)^2 \prod (\mu_2 - \mu_i)^2 \prod (\mu_i - \mu_j).$$

We next let $\mu_2 = \mu_0 + h$ and the third row is in the form (11). Then we subtract the first row and h times the second row from this third row, divide both sides by h^2, and take the limit as h approaches zero. Thus we get

$$\frac{1}{2}\begin{vmatrix} 1 & \mu_0 + 1 & (\mu_0 + 1)^2 & \cdots & (\mu_0 + 1)^{m-1} \\ 0 & 1 & 2(\mu_0 + 1) & \cdots & (m-1)(\mu_0 + 1)^{m-2} \\ 0 & 0 & 2\cdot 1 & \cdots & (m-1)(m-2)(\mu_0 + 1)^{m-3} \\ \cdots & & & & \end{vmatrix}$$

$$= \prod (\mu_0 - \mu_i)^3 \prod (\mu_i - \mu_j).$$

Since the determinant has been expressed as a product of differences of distinct numbers, it is not zero. It should be clear how the cases of higher multiplicity are treated and also how one could successively deal with the case of two multiple roots, three multiple roots and so forth.

Thus, in every case we have that the solutions previously obtained are linearly independent. If we have complex roots, we write the determinant in the form

$$\begin{vmatrix} 1 & \mu + 1 & \cdots & (\mu + 1)^{m-1} \\ 1 & \bar{\mu} + 1 & \cdots & (\bar{\mu} + 1)^{m-1} \\ \cdots & & & \end{vmatrix}$$

i.e. every line containing a complex number is followed by the corresponding line for the complex conjugate. Now a determinant in the form

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ \overline{A}_1 & \overline{A}_2 & \overline{A}_3 & \overline{A}_4 \\ B_1 & B_2 & B_3 & B_4 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_3 & \overline{B}_4 \end{vmatrix}$$

can be converted into one whose first row consists of the real parts of the A_i and the second row the imaginary parts, and whose third and fourth rows depend in the corresponding manner on the B_i, by multiplying on the left by the determinant

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{-i}{2} & \frac{i}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{-i}{2} & \frac{i}{2} \end{vmatrix} = -1/4.$$

Similarly, the determinant

$$\begin{vmatrix} 1 & \mu + 1 & \cdots & (\mu + 1)^{m-1} \\ 1 & \bar{\mu} + 1 & \cdots & (\bar{\mu} + 1)^{m-1} \\ \cdots & & & \end{vmatrix}$$

can be converted into the determinant of the real solutions we wish to use by multiplying by a non-zero determinant. Thus the lemma is established.

COROLLARY. *For the solutions obtained in Lemma A, we can suppose that the first determinant is not zero.*

LEMMA B. *Under the hypothesis of Lemma A, every solution*

$$\{x(0), x(1), \cdots\}$$

of (6B′) *is a linear combination of the m linearly independent solutions obtained in Lemma A.*

PROOF. From (6B′) we can infer that there are constants c_i such that for p greater than or equal to m

$$x(p) = \sum_{j=1}^{m} c_j x(p - j)$$

since the coefficient of $\Delta^m x$ is 1. We consider the matrix,

$$\begin{Bmatrix} x_1(0) & x_1(1) & \cdots \\ \cdots & & \\ x_m(0) & x_m(1) & \cdots \\ x(0) & x(1) & \cdots \end{Bmatrix}.$$

The above result shows that the $(m + 1)$-th column of this matrix is a linear combination of the first m columns. The $(m + 2)$-th column is a linear combination of 2, 3, $\cdots$, $m + 1$ columns, but since the $(m +)$-th column is a linear combination of the columns 1 to m it follows that the $(m + 2)$-th column is also. The above result can be used to show inductively that every column is a linear combination of the first m columns and this implies that the rank of the matrix is less than or equal to m. However, the preceding corollary now permits us to infer that the rank is m.

For p greater than m, let us consider the finite matrix obtained by ignoring the columns for which t is greater than p. This finite matrix is of rank m. From the fact that the determinant of order m in the upper left-hand column is not zero, we can infer that the last row is a linear combination of the first m rows with coefficients which are determined solely by the first m columns, i.e. we have

$$x(t) = \sum_{j=1}^{n} c_j x_j(t) \tag{12}$$

for $t = 0, 1, \cdots , p$. But since the first m of these equations are adequate to determine c_j the latter do not depend on p. Since p can be arbitrarily large, (12) holds for every p.

SOLUTION OF THE DIFFERENTIAL EQUATIONS OF MOTION OF A PROJECTILE IN A MEDIUM OF QUASI-NEWTONIAN RESISTANCE*

BY

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1. Introduction. An analytic solution for the basic differential equations of exterior ballistics has never been found. These equations express the path of a projectile in a retarding medium such as air. In practice these equations are solved by a method of successive approximations for each new set of constants or initial conditions. This is a very laborious process.

The principal difficulty is the complexity of the law of air resistance. Several solutions have been found on the basis of simple retardation laws. These, however, are not the true laws of resistance encountered by a projectile, as shown by experiment. The only solution which is of any practical value to ballisticians is that found by John Bernoulli for the Newtonian law of air resistance, $R = KV^2$. This is used in many cases where there are no accurate experimental data available.

Although resistance encountered by a projectile never behaves exactly according to Newton's law, there are many instances where the deviation from this law is "small". This is true for projectiles moving with velocities less than that of sound. In this paper a solution is obtained for the differential equations of motion of a body in a medium in which the resistance law is almost Newtonian, or quasi-Newtonian. This is done by making use of the theory of differential variations which takes into account differentials of the first order only, neglecting differentials of higher order.

In an illustrative example in the text the author works out a trajectory of a shell with an initial velocity of 300 m/s (the velocity of sound is approximately 340 m/s) and an angle of elevation of 60°, by both the new method and the standard method of successive approximations. The ranges obtained by the two methods are identical to the nearest meter, each giving a range of 4240 meters. Since the initial velocity in the illustrative problem is almost that of sound and the angle of elevation is rather large it may be expected that this method will give accurate results for all trajectories with initial velocities below sound. It should be pointed out that the variation of density with altitude is also taken into account.

The derivation of explicit expressions is confined in this paper to the Newtonian law of resistance because of its practical usefulness. However, it is apparent from subsequent discussions that the methods used are quite general, and are applicable with slight modifications to any resistance law for which a solution to the equations of motion can be obtained in finite form.

2. Resistance of a moving projectile. If a body is moving in a medium other than a vacuum, it is retarded according to some complex law which is related to the density of the medium, the size and shape of the moving body, and its velocity. An exact law which governs the amount of retardation, or deceleration has not yet been found. The motion is influenced by a number of factors which are too difficult for a complete theoretical analysis. For instance, when a projectile moving in air exceeds the velocity

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of sound, it produces a pattern of shock waves which use up its energy and greatly reduce its velocity. Experimental curves are now available which give the retardation for several shapes of projectiles and for varying velocities. From these curves it is apparent that the velocity of sound acts as a dividing line, at which the curve invariably experiences a steep and almost discontinuous rise due to the shock waves propagated by the moving body. The subsonic region is a region of quasi-Newtonian resistance to which the methods of this paper are applicable. In this region the law of resistance governing the moving body deviates from Newton's quadratic law of resistance, $R = KV^2$, by a "small" amount.

3. The equations of motion. For simplicity we will consider a projectile moving in a vertical plane, acted upon by the force of gravity mg (where m is the mass of the projectile) in a downward direction, and by a retardation force mR in a direction opposite to that of its motion. Let us fix a set of axes in this plane, y being the vertical axis and x the horizontal axis. The differential equations of motion may then be written in the form

$$\begin{aligned} \frac{d^2x}{dt^2} &= -E\frac{dx}{dt} \\ \frac{d^2y}{dt^2} &= -E\frac{dy}{dt} - g \end{aligned} \tag{1}$$

where $E = R/V$. In the general ballistic problem E is a function of y, the altitude, as well as of V, the velocity. E is usually written equal to $KG(V)H(y)$, where K is a constant characteristic of the projectile, $G(V)$ is a function of the velocity alone, and $H(y)$ is the density of the air aloft and is a function of y alone.

4. Solutions of Bernoulli and D'Alembert.[1] In 1719 John Bernoulli had obtained the solution of the above set of differential equations on the simplifying assumption that $R = KV^n$. Later, in 1744 d'Alembert integrated the same equations for the slightly more general resistance law $R = a + KV^n$. We are particularly interested in the special case $R = KV^2$, but since it does not present any additional difficulties we shall repeat here the derivation of the more general solution due to d'Alembert. In Cartesian coordinates $dx/dt = V\cos\theta$ and $dy/dt = V\sin\theta$, where θ is the angle between the tangent at any point of the trajectory and the x-axis. Equations (1) may then be written

$$\begin{aligned} \frac{d}{dt}(V\cos\theta) &= -(a + KV^n)\cos\theta \\ \frac{d}{dt}(V\sin\theta) &= -(a + KV^n)\sin\theta - g. \end{aligned} \tag{2}$$

Differentiating the left-hand side of Eqs. (2), then multiplying by $\sin\theta$ and $\cos\theta$, respectively, and subtracting, we obtain

$$\frac{dt}{d\theta} = -\frac{V}{g\cos\theta}. \tag{3}$$

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[1]K. J. Cranz and K. Becker, *Exterior ballistics* (Vol. I of *Handbook of Ballistics*), His Majesty's Stationery Off., London, 1921, Ch. IV.

It follows immediately that

$$\frac{dx}{d\theta} = \frac{dx}{dt}\frac{dt}{d\theta} = -\frac{V^2}{g} \tag{4}$$

$$\frac{dy}{d\theta} = \frac{dy}{dt}\frac{dt}{d\theta} = -\frac{V^2}{g}\tan\theta \tag{5}$$

$$\frac{ds}{d\theta} = \frac{ds}{dt}\frac{dt}{d\theta} = -\frac{V^2}{g\cos\theta}. \tag{6}$$

Substituting Eq. (3) into Eq. (1) and expanding the left-hand side, we also obtain the equation

$$g\cos\theta(V^{-n-1})dV - (a + g\sin\theta)V^{-n}d\theta = Kd\theta. \tag{7}$$

This equation may readily be integrated, giving an expression for V in terms of θ,

$$\begin{aligned} V^{-n} &= \cos^n\theta \exp\left[-ng^{-1}\int a(\cos\theta)^{-1}\,d\theta\right] \\ &\times \left\{-\frac{n}{g}\int \frac{K\exp\left[ng^{-1}\int a(\cos\theta)^{-1}\,d\theta\right]}{\cos^{n+1}\theta}\,d\theta + C\right\}. \end{aligned} \tag{8}$$

For the case $a = 0$, $n = 2$ and the initial conditions $V = V_0$ and $\theta = \phi$ we obtain

$$V^2 = \frac{g}{2K}\frac{1}{\cos^2\theta[C - f(\theta)]} \tag{9}$$

$$t = -\frac{1}{(2gK)^{1/2}}\int_\phi^\theta \frac{d\theta}{\cos^2\theta[C - f(\theta)]^{1/2}} \tag{10}$$

$$x = -\frac{1}{2K}\int_\phi^\theta \frac{d\theta}{\cos^2\theta[C - f(\theta)]} \tag{11}$$

$$y = -\frac{1}{2K}\int_\phi^\theta \frac{\tan\theta\,d\theta}{\cos^2\theta[C - f(\theta)]} \tag{12}$$

$$s = \frac{1}{2K}\log\frac{C - f(\theta)}{C - f(\phi)} \tag{13}$$

where

$$C = \frac{g}{2K(V_0\cos\phi)^2} + f(\phi)$$

and

$$f(\theta) = \int \sec^3\theta\,d\theta = \tfrac{1}{2}[\tan\theta\sec\theta + \log(\tan\theta + \sec\theta)].$$

5. Variational equations.[2] The theory of differential variations was developed during the first world war by G. A. Bliss, F. R. Moulton and T. H. Gronwall for the purpose

[2] F. R. Moulton, *New methods in exterior ballistics*, University of Chicago Press, 1926, Ch. IV.

of finding corrections to a trajectory due to non-standard ballistic conditions. A set of linear differential equations was derived which had to be solved in order to obtain these corrections. This was carried out by numerical methods, in a manner similar to that used in the solution of the equations of motion.

For a given set of initial conditions

$$\begin{aligned} x(0) &= 0, \qquad & y(0) &= 0, \\ x'(0) &= x_0', \qquad & y'(0) &= y_0', \end{aligned} \tag{14}$$

Eqs. (1) have a definite solution. In terms of t as independent variable, let this solution be represented by the functions

$$\begin{aligned} x &= \Phi(t), \qquad & y &= \Psi(t), \\ x' &= \Phi'(t), \qquad & y' &= \Psi'(t). \end{aligned} \tag{15}$$

We now introduce two types of perturbations in the original equations (1): we introduce small perturbations in the normal forces acting upon the projectile, thus changing Eqs. (1) to the form

$$\begin{aligned} \frac{d^2x}{dt^2} &= -E\frac{dx}{dt} + X \\ \frac{d^2y}{dt^2} &= -E\frac{dy}{dt} + Y - g \end{aligned} \tag{16}$$

where X and Y are the components of acceleration due to these forces, in the x and y directions, respectively; at the same time we change the initial conditions given in (14) by the addition of small increments α and β to the values of x_0' and y_0', respectively. The new initial conditions thus become

$$\begin{aligned} x(0) &= 0, \qquad & y(0) &= 0, \\ x'(0) &= x_0' + \alpha, \qquad & y'(0) &= y_0' + \beta. \end{aligned} \tag{17}$$

The solution to the new set of differential equations (16) with initial conditions (17) may be written in the form

$$\begin{aligned} x &= \Phi(t) + \xi(t), \qquad & y &= \Psi(t) + \eta(t), \\ x' &= \Phi'(t) + \xi'(t), \qquad & y' &= \Psi'(t) + \eta'(t), \end{aligned} \tag{18}$$

where $\xi(t)$ and $\eta(t)$ are small compared with $\Phi(t)$ and $\Psi(t)$.

Let us now make the assumption that $E = R/V$ is a function of x' and y' but not of y; and that it may be expanded as a power series in terms $\xi' = x' - \Phi$ and $\eta' = y' - \Psi'$. Substituting (18) into (1) and neglecting all differentials of higher order than the first, we obtain

$$\begin{aligned} \frac{d^2\Phi}{dt^2} + \frac{d^2\xi}{dt^2} &= -\left[E_0 + \frac{\partial E_0}{\partial x'}\xi' + \frac{\partial E_0}{\partial y'}\eta'\right](\Phi' + \xi') + X \\ \frac{d^2\Psi}{dt^2} + \frac{d^2\eta}{dt^2} &= -\left[E_0 + \frac{\partial E_0}{\partial x'}\xi' + \frac{\partial E_0}{\partial y'}\eta'\right](\Psi' + \eta') + Y - g \end{aligned} \tag{19}$$

where E_0 is evaluated at Φ', Ψ'. Since (15) is a solution of Eqs. (1) it follows that

$$\frac{d^2\Phi}{dt^2} = -E_0\Phi',$$
$$\frac{d^2\Psi}{dt^2} = -E_0\Psi' - g. \tag{20}$$

Hence, substituting into (19) and neglecting quadratic terms in ξ', η' we obtain

$$\frac{d^2\xi}{dt^2} = -E_0\xi' - \Phi'\left[\frac{\partial E_0}{\partial x'}\xi' + \frac{\partial E_0}{\partial y'}\eta'\right] + X$$
$$\frac{d^2\eta}{dt^2} = -E_0\eta' - \Psi'\left[\frac{\partial E_0}{\partial x'}\xi' + \frac{\partial E_0}{\partial y'}\eta'\right] + Y \tag{21}$$

or,

$$\frac{d^2\xi}{dt^2} = -\left(E_0 + \Phi'\frac{\partial E_0}{\partial x'}\right)\xi' - \Phi'\frac{\partial E_0}{\partial y'}\eta' + X$$
$$\frac{d^2\eta}{dt^2} = -\Psi'\frac{\partial E_0}{\partial x'}\xi' - \left(E_0 + \Psi'\frac{\partial E_0}{\partial y'}\right)\eta' + Y. \tag{22}$$

Setting $E = KG(V)$ (i.e. $H(y) \equiv 1$), we get

$$\frac{\partial E_0}{\partial x'} = K\frac{\partial G}{\partial V}\frac{\Phi'}{V} = \frac{E_0}{VG}\Phi'\frac{\partial G}{\partial V},$$
$$\frac{\partial E_0}{\partial y'} = K\frac{\partial G}{\partial V}\frac{\Psi'}{V} = \frac{E_0}{VG}\Psi'\frac{\partial G}{\partial V}. \tag{23}$$

Finally, substituting (23) into (22),

$$\frac{d^2\xi}{dt^2} = \frac{d\xi'}{dt} = P_1\xi' + P_2\eta' + X$$
$$\frac{d^2\eta}{dt^2} = \frac{d\eta'}{dt} = Q_1\xi' + Q_2\eta' + Y \tag{24}$$

where

$$P_1 = -\left(E_0 + \Phi'^2\frac{E_0}{VG}\frac{\partial G}{\partial V}\right),$$
$$P_2 = Q_1 = \Phi'\Psi'\frac{E_0}{VG}\frac{\partial G}{\partial V},$$
$$Q_2 = -\left(E_0 + \Psi'^2\frac{E_0}{VG}\frac{\partial G}{\partial V}\right).$$

This set of equations may also be written in the form

$$
\begin{aligned}
\frac{d\xi}{dt} &= 0 + \xi' \quad + 0 + 0 \\
\frac{d\xi'}{dt} &= 0 + P_1\xi' + 0 + P_2\eta' + X \\
\frac{d\eta}{dt} &= 0 + 0 \quad + 0 + \eta' \\
\frac{d\eta'}{dt} &= 0 + Q_1\xi' + 0 + Q_2\eta' + Y.
\end{aligned} \tag{25}
$$

The quantities E_0, Φ', $\partial G/\partial V$ etc. in the expressions for P_1, P_2, Q_1, Q_2 are defined for the unperturbed trajectory, which we shall assume has been solved. Consequently Eqs. (24) are a set of linear differential equations, the solution to which will give the quantities ξ, ξ', η, η' which represent the increments in x, x', y, y' due to a change α, β in the initial conditions and the perturbation forces X, Y.

6. Solution of equations for perturbed initial conditions. We shall first obtain a solution to Eqs. (24) for the case $X = Y = 0$. Under these conditions Eqs. (24) are reduced to a system of homogenous linear equations.

Let us assume that we have found two sets of solutions:

$$\xi'(t) = \xi_1'(t)$$

$$\eta'(t) = \eta_1'(t)$$

for the initial conditions

$$\xi'(0) = \xi_1'(0)$$

$$\eta'(0) = \eta_1'(0),$$

and

$$\xi'(t) = \xi_2'(t)$$

$$\eta'(t) = \eta_2'(t)$$

for the initial conditions

$$\xi'(0) = \xi_2'(0)$$

$$\eta'(0) = \eta_2'(0).$$

It follows from the theory of linear differential equations that all solutions of Eqs. (24) may then be written in the form

$$
\begin{aligned}
\xi'(t) &= C_1\xi_1'(t) + C_2\xi_2'(t), \\
\eta'(t) &= C_1\eta_1'(t) + C_2\eta_2'(t),
\end{aligned} \tag{26}
$$

where C_1, C_2 are constants, provided the determinant

$$D(t) = \begin{vmatrix} \xi_1'(t) & \xi_2'(t) \\ \eta_1'(t) & \eta_2'(t) \end{vmatrix} \neq 0. \tag{27}$$

It is also known that the value of this determinant is given by

$$D(t) = D(0) \exp\left[\int_0^t (P_1 + Q_2)\,dt\right]. \tag{28}$$

For the quadratic law of air resistance $E = KG(V) = KV$. Equations (25) then reduce to

$$\begin{aligned} \frac{d\xi'}{dt} &= -KV\left(1 + \frac{\Phi'^2}{V^2}\right)\xi' - \frac{K\Phi'\Psi'}{V}\eta', \\ \frac{d\eta'}{dt} &= -\frac{K\Phi'\Psi'}{V}\xi' - KV\left(1 + \frac{\Psi'^2}{V^2}\right)\eta'. \end{aligned} \tag{29}$$

These may also be written in the form

$$\begin{aligned} \frac{d\xi'}{dt} &= -KV(1 + \cos^2\theta)\xi' - KV\sin\theta\cos\theta\eta' \\ \frac{d\eta'}{dt} &= -KV\sin\theta\cos\theta\xi' - KV(1 + \sin^2\theta)\eta' \end{aligned} \tag{30}$$

where

$$V = \left(\frac{g}{K}\right)^{1/2} \frac{1}{\cos\theta[2C - 2f(\theta)]^{1/2}} \qquad \text{(Eq. 9)}.$$

In order to solve Eqs. (30) it is necessary to find a set of two particular solutions to this system. Such a set of solutions is in general furnished by the derivatives of the solution to Eqs. (1) with respect to the parameters introduced by the initial values. Let the solution to Eqs. (1) be given by

$$x' = \Phi'(t, x_0', y_0'), \qquad y' = \Psi'(t, x_0', y_0');$$

then

$$\xi' = \frac{\partial\Phi'}{\partial x_0'}, \qquad \eta' = \frac{\partial\Psi'}{\partial x_0'} \qquad \text{and} \qquad \xi' = \frac{\partial\Phi'}{\partial y_0'}, \qquad \eta' = \frac{\partial\Psi'}{\partial y_0'}$$

satisfy Eqs. (30). Furthermore, since t is not contained explicitly in Eqs. (1), it may be easily verified that

$$\xi' = \frac{d\Phi'}{dt}, \qquad \eta' = \frac{d\Psi'}{dt}$$

is also a solution of Eqs. (30). For convenience, we choose the latter solution and also the derivatives of Φ', Ψ' with respect to the parameter C. These are given in (31) and (32) below:

$$\begin{aligned} \xi_1'(t) &= \frac{1}{\cos\theta[2C - 2f(\theta)]}, \\ \eta_1'(t) &= \frac{\tan\theta}{\cos\theta[2C - 2f(\theta)]} + 1, \end{aligned} \tag{31}$$

with the initial conditions

$$\xi_1'(0) = \frac{KV_0^2 \cos \phi}{g},$$

$$\eta_1'(0) = \frac{KV_0^2 \sin \phi}{g} + 1;$$

and

$$\begin{aligned} \xi_2'(t) &= \xi_1'(t) \int_\phi^\theta \frac{d\theta}{\cos^2 \theta[2C - 2f(\theta)]^{3/2}} - \frac{1}{[2C - 2f(\theta)]^{3/2}}, \\ \eta_2'(t) &= \eta_1'(t) \int_\phi^\theta \frac{d\theta}{\cos^2 \theta[2C - 2f(\theta)]^{3/2}} - \frac{\tan \theta}{[2C - 2f(\theta)]^{3/2}}, \end{aligned} \tag{32}$$

with the initial conditions

$$\xi_2'(0) = -\left(\frac{K}{g}\right)^{3/2} (V_0 \cos \phi)^3,$$

$$\eta_2'(0) = -\left(\frac{K}{g}\right)^{3/2} (V_0 \cos \phi)^3 \tan \phi.$$

We will now evaluate the determinant $D(t)$. We note that

$$D(0) = \begin{vmatrix} \dfrac{1}{g} KV_0^2 \cos \phi & -\left(\dfrac{K}{g}\right)^{3/2} (V_0 \cos \phi)^3 \\ \dfrac{1}{g} KV_0^2 \sin \phi + 1 & -\left(\dfrac{K}{g}\right)^{3/2} (V_0 \cos \phi)^3 \tan \phi \end{vmatrix} = \left(\frac{K}{g}\right)^{3/2} (V_0 \cos \phi)^3.$$

On the other hand

$$\exp\left[\int_0^t (P_1 + Q_2)\, dt\right] = \exp\left(-3K \int_0^t V\, dt\right) = \left(\frac{g}{K}\right)^{3/2} (V_0 \cos \phi)^{-3} [2C - 2f(\theta)]^{-3/2}.$$

Hence,

$$D(t) = D(0) \exp\left[\int_0^t (P_1 + Q_2)\, dt\right] = [2C - 2f(\theta)]^{-3/2}.$$

It is apparent that $D(t)$ is not equal to zero, except when $\phi = 90°$. This case will be discussed separately in Sec. 12.

7. Solution of equations for perturbed forces. We are now ready to solve the set of non-homogeneous equations (24). We shall assume that the solution has the same form as (26) with the exception that C_1, C_2 are now functions of t rather than constants; namely,

$$\begin{aligned} \xi'(t) &= C_1(t)\xi_1'(t) + C_2(t)\xi_2'(t), \\ \eta'(t) &= C_1(t)\eta_1'(t) + C_2(t)\eta_2'(t). \end{aligned} \tag{33}$$

Differentiating (33), and remembering that (33) with C_1, C_2 constant is a solution to (24) when $X = Y = 0$, we obtain

$$\xi_1'(t)\frac{dC_1}{dt} + \xi_2'(t)\frac{dC_2}{dt} = X,$$
$$\eta_1'(t)\frac{dC_1}{dt} + \eta_2'(t)\frac{dC_2}{dt} = Y. \tag{34}$$

Hence,

$$\frac{dC_1}{dt} = D^{-1}(t)\begin{vmatrix} X & \xi_2'(t) \\ Y & \eta_2'(t) \end{vmatrix} = f_1(t)$$

and

$$\frac{dC_2}{dt} = D^{-1}(t)\begin{vmatrix} \xi_1'(t) & X \\ \eta_1'(t) & Y \end{vmatrix} = f_2(t). \tag{35}$$

Integrating we obtain

$$C_1(t) = B_1 + \int_0^t f_1(t)\,dt = B_1 + F_1(t)$$
$$C_2(t) = B_2 + \int_0^t f_2(t)\,dt = B_2 + F_2(t) \tag{36}$$

where B_1, B_2 are constants which may be determined from the initial conditions. Substituting these values into (33) we finally obtain a solution to Eqs. (24),

$$\xi'(t) = F_1(t)\xi_1'(t) + F_2(t)\xi_2'(t) + B_1\xi_1'(t) + B_2\xi_2'(t),$$
$$\eta'(t) = F_1(t)\eta_1'(t) + F_2(t)\eta_2'(t) + B_1\eta_1'(t) + B_2\eta_2'(t). \tag{37}$$

For normal initial conditions $B_1 = B_2 = 0$, hence

$$\xi'(t) = F_1(t)\xi_1'(t) + F_2(t)\xi_2'(t),$$
$$\eta'(t) = F_1(t)\eta_1'(t) + F_2(t)\eta_2'(t). \tag{38}$$

8. Quasi-Neutonian resistance. Equations (1) express the path of a projectile under the influence of a law of air resistance $E = R/V$. If the resistance is slightly perturbed, so that only differentials of the first order must be considered, while differentials of the second or higher order may be neglected, the resistance is then given by $E + \Delta E$. When this expression is substituted into Eqs. (1), Eqs. (16) are obtained with

$$X = -\Delta EV\cos\theta,$$
$$Y = -\Delta EV\sin\theta.$$

It follows that for the case of quasi-Newtonian resistance Eq. (35) may be written in the form

$$f_1(t) = -\Delta E \begin{vmatrix} \cos\theta & \xi_2' \\ \sin\theta & \eta_2' \end{vmatrix} V[2C - 2f(\theta)]^{3/2},$$

$$f_2(t) = -\Delta E \begin{vmatrix} \xi_1' & \cos\theta \\ \eta_1' & \sin\theta \end{vmatrix} V[2C - 2f(\theta)]^{3/2}. \tag{39}$$

Consequently,

$$F_1 = \int_\phi^\theta \frac{\Delta E}{K} \begin{vmatrix} \cos\theta & \xi_2' \\ \sin\theta & \eta_2' \end{vmatrix} \frac{[2C - 2f(\theta)]^{1/2}}{\cos^3\theta}\, d\theta,$$

$$F_2 = \int_\phi^\theta \frac{\Delta E}{K} \begin{vmatrix} \xi_1' & \cos\theta \\ \eta_1' & \sin\theta \end{vmatrix} \frac{[2C - 2f(\theta)]^{1/2}}{\cos^3\theta}\, d\theta. \tag{40}$$

We now may write expressions for the increments

$$\xi = -\int_\phi^\theta \frac{(F_1\xi_1' + F_2\xi_2')\, d\theta}{(gK)^{1/2}\cos^2\theta[2C - 2f(\theta)]^{1/2}}$$

$$\xi' = \frac{d\xi}{dt} = F_1\xi_1' + F_2\xi_2'$$

$$\eta = -\int_\phi^\theta \frac{(F_1\eta_1' + F_2\eta_2')\, d\theta}{(gK)^{1/2}\cos^2\theta[2C - 2f(\theta)]^{1/2}} \tag{41}$$

$$\eta' = \frac{d\eta}{dt} = F_1\eta_1' + F_2\eta_2'.$$

Equations (10), (11), (12), and (41) thus give the complete solution for a trajectory with quasi-Newtonian resistance. The procedure for carrying out a solution consists of several steps. First we solve Eqs. (10), (11), and (12), using a value of K that would give as closely as possible a true estimate of the resistance function. After this has been completed the function $\Delta E = E - KV$ is computed. ($E = KG(V)H(y)$ is the true resistance function, and may in practice also include the variation in density due to altitude.) Then the corrections ξ, ξ', η, η' due to a change ΔE in the resistance function are computed from Eqs. (41). The true values of t, x, x', y, y' which we shall denote by $\bar{t}$, $\bar{x}$, $\bar{x}'$, $\bar{y}$, $\bar{y}'$ are then given by the relations

$$\begin{aligned} \bar{t} &= t \\ \bar{x} &= x + \xi \\ \bar{x}' &= x' + \xi' \\ \bar{y} &= y + \eta \\ \bar{y}' &= y' + \eta'. \end{aligned} \tag{42}$$

In order to test the applicability of this method to practical ballistic problems a portion of a trajectory based on the standard Army resistance function G_2 was calculated by the usual method of successive approximations. The initial values were taken as follows:

Initial velocity $= 300\ m/s$,
Ballistic coefficient $= 1$,
Angle of elevation $= 60°$.

The solution obtained is compared in the table below with the solution by the method proposed in this paper. There seems to be perfect agreement between the two results, as far as the computation was carried.

Successive Approximations			New Method	
t sec.	$\bar{x}$ meters	$\bar{y}$ meters	$\bar{x}$ meters	$\bar{y}$ meters
0.00	0.0	0.0	0.0	0.0
1.00	146.8	249.5	146.8	249.5
2.00	287.9	479.6	287.8	479.5
3.00	424.0	692.0	424.0	691.9
4.00	555.8	888.0	555.8	887.9
5.00	683.8	1068.5	683.7	1068.4
6.00	808.2	1234.5	808.1	1234.4
7.00	929.6	1386.7	929.5	1386.6
8.00	1048.1	1525.7	1048.0	1525.5

9. The adjoint system of equations. In many ballistic problems only the terminal conditions of the trajectory, such as the range of the projectile when it strikes the ground, are required. In that event it is not necessary to obtain the path of the complete trajectory. By solving the system of equations adjoint to (25) it is possible to obtain directly the terminal values and thus to eliminate a large percentage of the computational work. In addition to this gain the method of the adjoint system yields expressions that can be used to find the effects upon the normal trajectory due to non-standard conditions of the air, such as wind or temperature variations. There is a third advantage to the computer in this method in that the final results are given for equal intervals of θ, which is not true in the case of the formulas derived in the preceding section.

The set of equations

$$\begin{aligned}
-\frac{d\rho}{dt} &= 0 + 0 \quad\;\; + 0 + 0 \\
-\frac{d\rho'}{dt} &= \rho + P_1\rho' + 0 + Q_1\sigma' \\
-\frac{d\sigma}{dt} &= 0 + 0 \quad\;\; + 0 + 0 \\
-\frac{d\sigma'}{dt} &= 0 + P_2\rho' + \sigma + Q_2\sigma'
\end{aligned} \tag{43}$$

whose coefficients are the same as those of Eqs. (25) with rows and columns interchanged, is known as the system of equations adjoint to (25). If a solution to (43) is found for the variables $\rho, \rho', \sigma, \sigma'$ the quantities ξ, ξ', η, η' may then be expressed in terms of these variables. The relationship between these two sets of variables may be easily derived.

Multiplying Eqs. (25) by $\rho, \rho', \sigma, \sigma'$ respectively, and adding we obtain

$$\begin{aligned} &\rho \frac{d\xi}{dt} + \rho' \frac{d\xi'}{dt} + \sigma \frac{d\eta}{dt} + \sigma' \frac{d\eta'}{dt} \\ &\quad = \xi'(\rho + P_1\rho' + Q_1\sigma') + \eta'(\sigma + P_2\rho' + Q_2\sigma') + (\rho' X + \sigma' Y). \end{aligned} \tag{44}$$

Substituting Eqs. (43) we obtain

$$\rho \frac{d\xi}{dt} + \rho' \frac{d\xi'}{dt} + \sigma \frac{d\eta}{dt} + \sigma' \frac{d\eta'}{dt} + \xi \frac{d\rho}{dt} + \xi' \frac{d\rho'}{dt} + \eta \frac{d\sigma}{dt} + \eta' \frac{d\sigma'}{dt} = \rho' X + \sigma' Y \tag{45}$$

or

$$\frac{d}{dt}(\rho\xi + \rho'\xi' + \sigma\eta + \sigma'\eta') = \rho' X + \sigma' Y. \tag{46}$$

Finally,

$$\Big[\rho\xi + \rho'\xi' + \sigma\eta + \sigma'\eta'\Big]_0^T = \int_0^T (\rho' X + \sigma' Y)\, dt \tag{47}$$

where T is the time at the terminal point of the unperturbed or Newtonian trajectory.

10. Solution of the Adjoint System. It follows immediately from the first and third of Eqs. (43) that ρ = constant and that σ = constant. We shall denote these constants by $\bar{\rho}$ and $\bar{\sigma}$, respectively. The second and fourth equations may then be written

$$\begin{aligned} -\frac{d\rho'}{dt} &= \bar{\rho} + P_1\rho' + Q_1\sigma' \\ -\frac{d\sigma'}{dt} &= \bar{\sigma} + P_2\rho' + Q_2\sigma', \end{aligned} \tag{48}$$

or

$$\begin{aligned} -\frac{d\rho'}{dt} &= \bar{\rho} - \left(E_0 + \Phi'^2 \frac{E_0}{VG}\frac{\partial G}{\partial V}\right)\rho' - \Phi'\Psi' \frac{E_0}{VG}\frac{\partial G}{\partial V}\sigma', \\ -\frac{d\sigma'}{dt} &= \bar{\sigma} - \Phi'\Psi' \frac{E_0}{VG}\frac{\partial G}{\partial V}\rho' - \left(E_0 + \Psi'^2 \frac{E_0}{VG}\frac{\partial G}{\partial V}\right)\sigma'. \end{aligned} \tag{49}$$

It may be verified that one solution of this system is

$$\bar{\rho}_1\Phi' + \bar{\sigma}_1\Psi' + \rho_1'\Phi'' + \sigma_1'\Psi'' = -K_1 \qquad \text{(constant)}. \tag{50}$$

This follows from the fact that $\xi = \Phi'$, $\xi' = \Phi''$, $\eta = \Psi'$, $\eta' = \Psi''$ is a solution to Eqs. (24), and from (46) which reduces to

$$\frac{d}{dt}(\rho\xi + \rho'\xi' + \sigma\eta + \sigma'\eta') = 0 \qquad \text{for} \qquad X = 0, \qquad Y = 0.$$

Transposing the terms of Eq. (50) and dividing by Φ'' we obtain

$$-\rho_1' = \frac{\bar{\rho}_1\Phi' + \bar{\sigma}_1\Psi' + \sigma_1'\Psi'' + K_1}{\Phi''} \tag{51}$$

which may be used with the second equation of (49) to eliminate ρ_1'; thus

$$\begin{aligned} -\frac{d\sigma_1'}{dt} = \bar{\sigma}_1 + \Phi'\Psi' \frac{E_0}{VG}\frac{\partial G}{\partial V}\frac{\bar{\rho}_1\Phi' + \bar{\sigma}_1\Psi' + \sigma_1'\Psi'' + K_1}{\Phi''} \\ - \left(E_0 + \Psi'^2 \frac{E_0}{VG}\frac{\partial G}{\partial V}\right)\sigma_1' . \end{aligned} \tag{52}$$

Remembering that $\Phi' = x'$, $\Psi' = y'$ of the undisturbed or Newtonian trajectory, and that $\Phi'' = -E_0\Phi'$, the preceding equation reduces to a linear differential equation of the first order which may be solved immediately; namely

$$\frac{d\sigma_1'}{dt} + A(t)\sigma_1' + B(t) = 0 \tag{53}$$

where

$$A(t) = g\Psi' \frac{d \log G}{V\, dV} - E_0 ,$$

$$B(t) = \bar{\sigma}_1 - \Psi' \frac{d \log G}{V\, dV}(\bar{\rho}_1\Phi' + \bar{\sigma}_1\Psi' + K_1).$$

The solution of this equation is

$$\sigma_1' = \exp\left(-\int_0^t A(t)\, dt\right)\left[C_3 - \int_0^t B(t) \exp\left(\int_0^t A(t)\, dt\right) dt\right]. \tag{54}$$

The expression for ρ_1' is given by Eq. (51) above.

We shall want only the solution for the case $\sigma_1' = 0$ at $t = T$. Hence Eq. (54) may be written

$$\begin{aligned} \sigma_1'(t) = \exp\left(-\int_0^t A(t)\, dt\right) \\ \cdot\left[\int_0^T B(t) \exp\left(\int_0^t A(t)\, dt\right) dt - \int_0^t B(t) \exp\left(\int_0^t A(t)\, dt\right) dt\right]. \end{aligned} \tag{55}$$

We are particularly interested in the solution of the above equation for the special case $E = KG(V) = KV$; hence $d \log G/VdV = 1/V^2$. From Eqs. (3), (4), (5) and (9) it follows that,

$$A(t) = \frac{g \sin\theta - KV^2}{V}$$

and

$$\int_0^t A(t)\, dt = -\int_\phi^\theta \frac{g\sin\theta - KV^2}{g\cos\theta}\, d\theta = -\int_\phi^\theta \tan\theta\, d\theta + \frac{1}{2}\int_\phi^\theta \frac{d\theta}{\cos^3\theta[C - f(\theta)]} .$$

This may be integrated, giving the solution

$$\int_0^t A(t)\,dt = \log\left\{\frac{\cos\theta}{\cos\phi}\left[\frac{C - f(\phi)}{C - f(\theta)}\right]^{1/2}\right\};$$

and finally,

$$\exp\left(\int_0^t A(t)\,dt\right) = \frac{\cos\theta}{\cos\phi}\left[\frac{C - f(\phi)}{C - f(\theta)}\right]^{1/2} = \left(\frac{g}{K}\right)^{1/2}\frac{\cos\theta}{V_0\cos^2\phi[2C - 2f(\theta)]^{1/2}}\,. \tag{56}$$

Also,

$$B(t) = \bar{\sigma}_1 - \frac{\sin\theta}{V}(\bar{\rho}_1 V\cos\theta + \bar{\sigma}_1 V\sin\theta + K_1)$$

$$= \bar{\sigma}_1\cos^2\theta - \bar{\rho}_1\cos\theta\sin\theta - \frac{K_1\sin\theta}{V}\,.$$

Hence,

$$\int_0^t B(t)\exp\left(\int_0^t A(t)\,dt\right)dt = \frac{1}{KV_0\cos^2\phi}\int_\phi^\theta\left[\left(\frac{K}{g}\right)^{1/2}\frac{K_1\sin\theta}{[2C - 2f(\theta)]^{1/2}} - \frac{\bar{\sigma}_1\cos\theta - \bar{\rho}_1\cos\theta}{2C - 2f(\theta)}\right]d\theta.$$

Finally,

$$\sigma_1' = \frac{1}{(Kg)^{1/2}}\frac{[2C - 2f(\theta)]^{1/2}}{\cos\theta}\int_\theta^\omega\left[\left(\frac{K}{g}\right)^{1/2}\frac{K_1\sin\theta}{[2C - 2f(\theta)]^{1/2}} - \frac{\bar{\sigma}_1\cos\theta - \bar{\rho}_1\cos\theta}{2C - 2f(\theta)}\right]d\theta$$

$$\rho_1' = -\frac{\bar{\rho}_1\Phi' + \bar{\sigma}_1\Psi' + \sigma_1'\Psi'' + K_1}{\Phi''} \tag{57}$$

where ω is the angle at the terminal point of the trajectory.

11. Range and time of flight. As is well-known, the solution derived for the adjoint system of differential equations (43) may be used to obtain the range, the total time of flight or other terminal values of the perturbed trajectory. The quantities that are of greatest interest are the range (the horizontal distance from the muzzle of the gun to the point where the projectile strikes the ground) and the time of flight.

Let us suppose that Eqs. (10), (11), and (12) have been solved for a trajectory with given initial conditions. Let T be the time of flight to the ground, and X_T the range. These equations are based on the Newtonian resistance law, and the resulting trajectory is an approximation to the true trajectory. Since the true trajectory which obeys a quasi-Newtonian resistance law is not much different from the Newtonian trajectory we shall consider only differentials of the first order. In Fig. 1, X_T is the range of the Newtonian trajectory, $X_{T'}$ is the range of the quasi-Newtonian trajectory, P is the location of the projectile at time T, and $P0$ is a line normal to the x-axis. The change in horizontal range $\xi(T) = X_T0$; while $X_{T'}0 = -x'(T)\eta(T)/y'(T)$. Hence $X_TX_{T'}$ the total increment in range, is given by the quantity $\xi(T) - x'(T)\eta(T)/y'(T)$. Likewise, the increment in time of flight, which is the time the projectile moves from P to $X_{T'}$, is given by $-\eta(T)/y'(T)$.

In the preceding section we have found the solution to the adjoint system of differential equations. We may now assign initial conditions to ρ, ρ', σ, σ'. Let us first assign the set of initial conditions

$$\rho(T) = 1, \qquad \rho'(T) = 0, \qquad \sigma(T) = \frac{x'(T)}{y'(T)}, \qquad \sigma'(T) = 0. \tag{58}$$

Substituting these in Eq. (47) we obtain, since $\xi(0) = \eta(0) = 0$

$$\xi(T) - \frac{x'(T)}{y'(T)}\,\eta(T) = \rho'(0)\xi'(0) + \sigma'(0)\eta'(0) + \int_0^T (\rho'X + \sigma'Y)\,dt. \tag{59}$$

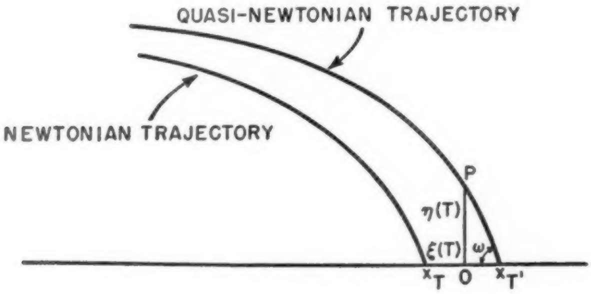

Fig. 1

It will be noticed that the left-hand side of Eq. (59) is the increment in range derived from geometrical considerations. Thus Eq. (59) gives an expression for the difference in range between Newtonian and quasi-Newtonian resistance in terms of the functions σ' and ρ' of the adjoint system. These are given by Eqs. (57) above.

In order to obtain the increment in time of flight we assign the initial conditions

$$\rho(T) = 0, \qquad \rho'(T) = 0, \qquad \sigma(T) = \frac{1}{y'(T)}, \qquad \sigma'(T) = 0. \tag{60}$$

These give, by substitution in (47)

$$-\frac{\eta(T)}{y'(T)} = \rho'(0)\xi'(0) + \sigma'(0)\eta'(0) + \int_0^T (\rho'X + \sigma'Y)\,dt, \tag{61}$$

which is an expression for the increment in time. The functions ρ' and σ' may again be obtained from Eqs. (57). It will be noted that the value of K_1 is not the same for range and time of flight but must be determined from Eq. (50) before solving Eqs. (57). For the increment in range K_1 is equal to zero, while for the increment in time of flight $K_1 = 1$.

A complete trajectory with the initial conditions used for the illustrative problem in Sec. 8 was computed by the standard method and by the method of the adjoint system. The ranges obtained by the two methods were identical to the nearest meter, each giving a range of 4240 meters. The actual values obtained were 4239.8 meters and 4240.2 meters. This may be considered complete agreement since the inaccuracy due

to the method of successive approximations is of the same magnitude as the discrepancy. For practical ballistic problems this is far greater accuracy than is usually desired.

12. Vertical trajectories. It was pointed out in Sec. 6 that the equations derived there for a trajectory with perturbed initial conditions do not hold for a vertical trajectory, since for such a trajectory $dx/dt = 0$. In this section we will derive simplified formulas for this special case. It will be of interest to note here that for the case of a vertical trajectory the solution of the equations of motion for Newtonian resistance with variation in air resistance due to an exponential law of density aloft is known. The solution may be given in terms of the function $Ei(z) = \int_{-\infty}^{z} (e^z/z)dz$.

In one dimension the equation of motion is

$$\frac{d^2y}{dt^2} = -E\frac{dy}{dt} - g. \tag{62}$$

Let the solution to this equation for the initial conditions

$$y(0) = 0$$

$$y'(0) = y_0'$$

be given by

$$y = \Psi(t)$$

$$y' = \Psi'(t).$$

Also, let the solution for non-standard initial conditions

$$y(0) = 0$$

$$y'(0) = y_0' + \beta$$

be given by

$$y = \Psi(t) + \eta(t)$$

$$y' = \Psi(t) + \eta'(t).$$

If we assume in addition that $E = R/V$ is a function of $y' = V$ alone and may be expanded in terms of $\eta' = (y' - \Psi')$, we may write

$$\frac{d^2y}{dt^2} + \frac{d^2\eta}{dt^2} = -\left[E_0 + \frac{dE_0}{dy'}\,\eta'\right](\Psi' + \eta') - g. \tag{63}$$

Substituting Eq. (62) into (63) and neglecting terms in η'^2 we obtain

$$\frac{d^2\eta}{dt^2} = -E_0\eta' - y'\frac{dE_0}{dy'}\,\eta'. \tag{64}$$

Since $y' = V$ and E can be written $E = KG(V)$, the above equation reduces to

$$\frac{d^2\eta}{dt^2} = \frac{d\eta'}{dt} = -\left[KG(V) + KV\frac{dG}{dV}\right]\eta'. \tag{65}$$

Hence

$$\eta' = C_4 \exp\left[-\int \left(KG(V) + KV\frac{dG}{dV}\right) dt\right]. \tag{66}$$

For a small abnormal force Y acting on the projectile, Eq. (62) may be written

$$\frac{d^2y}{dt^2} = -E\frac{dy}{dt} + Y - g. \tag{67}$$

For this case

$$\frac{d^2\eta}{dt^2} = \frac{d\eta'}{dt} = -\left[KG(V) + KV\frac{dG}{dV}\right]\eta' + Y \tag{68}$$

which is a linear differential equation. The solution is given by

$$\begin{aligned}\eta' = \exp&\left[-K\int\left(G(V) + V\frac{dG}{dV}\right) dt\right]\\ &\cdot\left\{C_5 + \int Y \exp\left[K\int\left(G(V) + V\frac{dG}{dV}\right) dt\right] dt\right\}.\end{aligned} \tag{69}$$

For the special case $G(V) = V$

$$\eta' = \exp(-2Ky)\left[C_5 + \int Y \exp(2Ky)\, dt\right]. \tag{70}$$

SOLUTION OF STEADY STATE TEMPERATURE PROBLEMS WITH THE AID OF A GENERALIZED FOURIER CONVOLUTION*[1]

BY

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1. Introduction. The purpose of this paper is to illustrate the use of the finite Fourier transformation and a generalized convolution in the solution of steady state boundary value problems. Solutions of some basic problems are given in terms of special functions introduced, and by means of a modified Duhamel formula other solutions in turn are expressed as functions of the basic solutions.

The finite sine and cosine transformation are defined by

$$S\{F(x)\} = \int_0^\pi F(x) \sin nx \, dx = f_s(n) \qquad (n = 1, 2, \cdots),$$

$$C\{F(x)\} = \int_0^\pi F(x) \cos nx \, dx = f_c(n) \qquad (n = 0, 1, 2, \cdots),$$

respectively. These transformations applied to the derivatives of $F(x)$ yield, for example, the following: $S\{F'(x)\} = -nC\{F(x)\}$, $S\{F''(x)\} = -n^2S\{F(x)\} + n[F(0) - (-1)^nF(\pi)]$, and $C\{F'(x)\} = nS\{F(x)\} - F(0) + (-1)^nF(\pi)$.

If $F(x)$ in $(-2\pi, 2\pi)$ and $G(x)$ in $(-\pi, \pi)$, are sectionally continuous functions, then the function

$$F(x)^*G(x) = \int_{-\pi}^{\pi} F(x - y)G(y) \, dy \tag{1}$$

is called the convolution of F and G in the interval $(-\pi, \pi)$.

If $F(x)$ is an odd and $G(x)$ is an even sectionally continuous function and if $F(x + 2\pi) = F(x)$, then the product of the transforms can be written in terms of the transform of the convolution, for example,[2]

$$S\{F(x)\}C\{G(x)\} = \tfrac{1}{2} S\{F(x)^*G(x)\}. \tag{2}$$

2. A generalized Fourier convolution. Let $F(x, y)$ be a sectionally continuous function of x and y in the square $0 \leq x \leq \pi$, $0 \leq y \leq \pi$. A generalized convolution $F^*(x)$ of $F(x, y)$ corresponding to the iterated finite sine transformation

$$S\{S\{F(x, y)\}\} = \int_0^\pi \int_0^\pi F(x, y) \sin nx \sin n'y \, dx \, dy$$

$$= \bar{f}(n, n') \qquad (n, n' = 1, 2, \cdots)$$

is defined by

$$F^*(x) = -\int_{-\pi}^{\pi} F_1(x - y, y) \, dy,$$

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[2]R. V. Churchill, Modern Operational Mathematics in Engineering, McGraw-Hill, 1944, p. 274-276.

where $F_1(x, y)$ is an odd periodic extension of F with respect to x and an odd extension with respect to y.

When $n' = n$, it can be shown[3] that

$$S\{S\{F(x, y)\}\} = \tfrac{1}{2} C\{F^*(x)\}.$$

In case $F(x, y) = F_1(x)G_1(y)$, the function $F^*(x)$ is the convolution (1) of F_1 and G_1.

3. Special functions. In this section two key functions are introduced as well as their sine and cosine transforms. In the sequel solutions of some basic boundary value problems are expressed in terms of these functions.

If $|r| < 1$, then $\log(1 + re^{i\theta}) = -\sum_{\nu=1}^{\infty} (-r)^\nu e^{i\nu\theta}/\nu$. Also if $R^2 = (1 + r\cos\theta)^2 + r^2\sin^2\theta$ and $\tan\Phi = r\sin\theta/(1 + r\cos\theta)$, then

$$\log(1 + re^{i\theta}) = \log Re^{i\Phi} = \log R + i\Phi \tag{5}$$

Equating the imaginary parts of (5), we get

$$\arctan\frac{r\sin\theta}{1 + r\cos\theta} = -\sum_{\nu=1}^{\infty}\frac{(-r)^\nu \sin\nu\theta}{\nu}.$$

When $|r| \leq 1$, the series is a Fourier sine series. The sine transforms of the function are, except for the normalizing factor $\pi/2$, the Fourier sine coefficients, i.e.,

$$S\left\{\arctan\frac{r\sin x}{1 + r\cos x}\right\} = -\frac{\pi}{2}\frac{(-1)^n r^n}{n}, \qquad |r| \leq 1,$$

where θ has been replaced by x. Since $(-1)^{n+1}f_s(n) = S\{F(\pi - x)\}$, then

$$S\left\{\arctan\frac{r\sin x}{1 - r\cos x}\right\} = \frac{\pi}{2}\frac{r^n}{n}.$$

Setting $r = e^{-z}$, the above formula becomes

$$S\left\{\arctan\frac{\sin x}{e^z - \cos x}\right\} = \frac{\pi}{2}\frac{e^{-nz}}{n}, \qquad z \geq 0. \tag{6}$$

Let $b_0(n, u)$ denote the function

$$\frac{\sinh nu}{n\sinh n\pi} = \sum_{\nu=0}^{\infty}\frac{1}{n}\{\exp\{-[(2\nu + 1)\pi - u]n\} - \exp\{-[(2\nu + 1)\pi + u]n\}\}. \tag{7}$$

Let the inverse sine transform of the function $b_0(n, u)$, $S^{-1}\{b_0(n, u)\}$, be $B_{01}(x, u)$, then according to formula (6), when $\pi - u \geq 0$,

$$\begin{aligned} B_{01}(x, u) &= \sum_{\nu=0}^{\infty}\left[\arctan\frac{\sin x}{\exp\{(2\nu + 1)\pi - u\} - \cos x}\right. \\ &\qquad \left. - \arctan\frac{\sin x}{\exp\{(2\nu + 1)\pi + u\} - \cos x}\right] \\ &= \sum_{\nu=0}^{\infty}\arctan\frac{\sin x\sinh u}{\cosh(2\nu + 1)\pi - \cos x\cosh u}. \end{aligned} \tag{8}$$

[3]A. W. Jacobson, A Generalized Convolution for the Finite Fourier Transformations, Thesis, University of Michigan, 1948, p. 9-12. In the sequel reference to this thesis will be marked [3].

Equating the real parts of (5), we have

$$\log \frac{1}{1 + 2r \cos x + r^2} = 2 \sum_{\nu=1}^{\infty} \frac{(-1)^\nu r^\nu}{\nu} \cos \nu x.$$

Hence the cosine transformation yields

$$C\left\{\log \frac{1}{1 + 2r \cos x + r^2}\right\} = \frac{\pi(-1)^n r^n}{n}, \qquad n = 1, 2, \cdots$$

$$= 0 \qquad n = 0$$

Since $(-1)^n f_c(n) = C\{F(\pi - x)\}$, and setting $r = e^{-z}$, we get when $z \geq 0$

$$C\left\{\log \frac{e^z}{2[\cosh z - \cos x]}\right\} = \frac{\pi e^{-nz}}{n}, \qquad n = 1, 2, \cdots \tag{9}$$

$$= 0 \qquad n = 0.$$

Let $b_1(n, u)$ denote the function

$$\frac{\cosh nu}{n \sinh n\pi} = \sum_{\nu=0}^{\infty} \frac{1}{n} \{\exp \{-[(2\nu + 1)\pi - u]n\} + \exp \{-[(2\nu + 1)\pi + u]n\}\}. \tag{10}$$

And let $B_{12}(x, u)$ denote the inverse cosine transform $C^{-1}\{b_1(n, u)\}$ of the function $b_1(n, u)$. When $\pi - u \geq 0$, according to formula (9), we get

$$B_{12}(x, u) = \frac{1}{\pi} \log \prod_{\nu=0}^{\infty} \frac{\exp \{2(2\nu + 1)\pi\}}{4[\cosh (2\nu\pi + \pi - u) - \cos x][\cosh (2\nu\pi + \pi + u) - \cos x]}. \tag{11}$$

4. Problems in two dimensions. Basic problems (A). Let $U_0(x, y)$ be the solution of the following steady state temperature problem in the region R: $0 \leq x \leq \pi$, $0 \leq y \leq \pi$:

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 0 \qquad \text{in} \qquad R,$$

$$U_0(+0, y) = U_0(\pi - 0, y) = 0, \qquad 0 < y < \pi, \tag{A}$$

$$U_0(x, +0) = \frac{\pi - x}{\pi}, \qquad U_0(x, \pi - 0) = 0, \qquad 0 < x < \pi.$$

The sine transformation of problem (A) with respect to x yields

$$\frac{d^2 u_0}{dy^2} - n^2 u_0(n, y) = 0,$$

$$u_0(n, 0) = \frac{1}{n}, \qquad u_0(n, \pi) = 0. \tag{A$'$}$$

The solution of the transformed problem is

$$u_0(n, y) = \frac{\sinh n(\pi - y)}{n \sinh n\pi}.$$

In the previous section, formula (7), this function was designated by the function $b_0(n, u)$, i.e.,

$$u_0(n, y) = b_0(n, \pi - y).$$

According to formula (8), its inverse sine transform is the function $B_{01}(x, \pi - y)$, so that

$$U_0(x, y) = B_{01}(x, \pi - y). \tag{12}$$

Problem (B)

$$\begin{aligned} &\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{in} \quad R \\ &U(+0, y) = U(\pi - 0, y) = 0, \qquad 0 < y < \pi, \\ &U(x, +0) = H(x), \quad U(x, \pi - 0) = 0, \qquad 0 < x < \pi. \end{aligned} \tag{B}$$

The transformed problem is

$$\begin{aligned} &\frac{d^2 u}{dy^2} - n^2 u(n, y) = 0, \\ &u(n, 0) = h(n), \quad u(n, \pi) = 0. \end{aligned} \tag{B'}$$

Upon multiplying the equations in problem (A′) by the function $nh(n)$ of the parameter, it is evident that the product $nh(n)u_0(n, y)$ is also a solution of problem (B′). If the solution of the latter is unique then

$$u(n, y) = nh(n)u_0(n, y).$$

Since $U_0(0, y) = U_0(\pi, y) = 0$, then $nS\{U(x, y)\} = C\{\partial/\partial x\, U_0(x, y)\}$. Hence

$$S\{U(x, y)\} = S\{H(x)\}nS\{U_0(x, y)\}.$$

becomes

$$S\{U(x, y)\} = S\{H(x)\}C\left\{\frac{\partial}{\partial x} U_0(x, y)\right\}.$$

The product of the transforms here can be written, according to formula (2), as the sine transform of the convolution of the two functions, and making the inverse transformation, there results

$$U(x, y) = \tfrac{1}{2}H(x)^* \frac{\partial}{\partial x} U_0(x, y).$$

Since $U_0(x, y) = B_{01}(x, \pi - y)$, equation (12), the last result becomes in terms of the convolution integral (1)

$$U(x, y) = \frac{1}{2}\int_{-\pi}^{\pi} H(x - \lambda) \frac{\partial}{\partial \lambda} B_{01}(\lambda, \pi - y)\, d\lambda.$$

In view of formula (11) $\partial/\partial\lambda\, B_{01}(\lambda, \pi - y)$ is an even function of λ, hence H is to be extended to $(-\pi, 0)$ as an odd function.

Problem (C).

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x, y) \qquad \text{in} \qquad R$$

$$U(+0, y) = U(\pi - 0, y) = 0, \qquad 0 < y < \pi \tag{C}$$

$$U(x, +0) = U(x, \pi - 0) = 0, \qquad 0 < x < \pi.$$

The sine transformation of this problem with respect to x yields

$$\frac{d^2u}{dy^2} - n^2 u(n, y) = f(n, y) \tag{C'}$$

$$u(n, 0) = u(n, \pi) = 0.$$

The solution of the transformed problem, in terms of the Green's function $g(n, y, \mu)$, is

$$u(n, y) = \int_0^\pi g(n, y, \mu) f(n, \mu)\, d\mu, \tag{13}$$

where

$$g(n, y, \mu) = \frac{\sinh n\mu \sinh n(\pi - y)}{n \sinh n\pi}, \qquad \mu \leq y$$

$$= \frac{\sinh ny \sinh n(\pi - \mu)}{n \sinh n\pi}, \qquad \mu \geq y,$$

or

$$g(n, y, \mu) = \frac{\cosh n(\pi - y - \mu)}{2n \sinh n\pi} - \frac{\cosh n(\pi - y + \mu)}{2n \sinh n\pi}, \qquad \mu \leq y$$

$$= \frac{\cosh n(\pi - y - \mu)}{2n \sinh n\pi} - \frac{\cosh n(\pi - \mu + y)}{2n \sinh n\pi}, \qquad \mu \geq y.$$

According to formula (10) this can be written

$$g(n, y, \mu) = \tfrac{1}{2} b_1(n, \pi - y - \mu) - \tfrac{1}{2} b_1(n, \pi - y + \mu), \qquad \mu \leq y$$

$$= \tfrac{1}{2} b_1(n, \pi - y - \mu) - \tfrac{1}{2} b_1(n, \pi - \mu + y), \qquad \mu \geq y.$$

The inverse cosine transform of $b_1(n, u)$ is, in view of formula (11), the function $B_{12}(x, u)$. Hence

$$G(x, y, \mu) = \tfrac{1}{2} B_{12}(x, \pi - y - \mu) - \tfrac{1}{2} B_{12}(x, \pi - y + \mu). \tag{14}$$

where

$$C^{-1}\{g(n, y, \mu)\} = G(x, y, \mu).$$

Writing solution (13) as

$$S\{U(x, y)\} = \int_0^\pi C\{G(x, y, \mu)\} S\{F(x, \mu)\}\, d\mu,$$

and replacing the product of the transforms by the sine transform of the convolution and making the inverse transformation, we obtain

$$U(x, y) = \frac{1}{2} \int_0^\pi G(x, y, \mu)^* F(x, \mu)\, d\mu,$$

or

$$U(x, y) = \frac{1}{2} \int_0^\pi \int_{-\pi}^\pi G(x - \lambda, y, \mu) F(\lambda, \mu)\, d\lambda\, d\mu.$$

Substituting for G from (14), there results

$$U(x, y) = \frac{1}{4} \int_0^\pi \int_{-\pi}^\pi [B_{12}(x - \lambda, \pi - y - \mu) - B_{12}(x - \lambda, \pi - y + \mu)] F(\lambda, \mu)\, d\lambda\, d\mu.$$

The function $B_{12}(x, u)$ is an even function of x, hence $H(x, y)$ must be extended to $(-\pi, 0)$ as an odd function of x.

The above solutions can be shown to satisfy the respective boundary value problems for a wide class of functions involved. See [3] p. 58-75.

5. Resolution of boundary value problems with the aid of a generalized Fourier convolution. Let $U(x, y)$ be the steady state temperature in the region R: $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $0 \leq z \leq \pi$, satisfying the following conditions:

$$\begin{aligned} &\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = F(x, y, z) \qquad \text{in} \qquad R, \\ &U(0, y, z) = H_1(y, z), \qquad U(\pi, y, z) = 0, \\ &U(x, 0, z) = H_3(x, y), \qquad U(x, \pi, z) = 0, \\ &U(x, y, 0) = H_5(x, y), \qquad U(x, y, \pi) = 0. \end{aligned} \tag{D}$$

Three of the boundary conditions have been written here as homogeneous since this involves no loss in generality. The sine transform of this problem with respect to x is

$$\begin{aligned} &-n^2 u(n, y, z) + nH_1(y, z) + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(n, y, z), \\ &u(n, 0, z) = h_3(n, z), \qquad u(n, \pi, z) = 0, \\ &u(n, y, 0) = h_5(n, y), \qquad u(n, y, \pi) = 0. \end{aligned}$$

Now let $V(x, x', y, z)$, depending on the parameter x' independent of x, y, z, be the solution of the following auxiliary problem:

$$\begin{aligned} &\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\pi - x}{\pi} F(x', y, z) \qquad \text{in} \qquad R, \\ &V(0, x', y, z) = \frac{\pi - x'}{\pi} H_1(y, z), \qquad V(\pi, x', y, z) = 0, \\ &V(x, x', 0, z) = \frac{\pi - x}{\pi} H_3(x', z), \qquad V(x, x', \pi, z) = 0, \\ &V(x, x', y, 0) = \frac{\pi - x}{\pi} H_5(x', y), \qquad V(x, x', y, \pi) = 0. \end{aligned} \tag{E}$$

The sine transform of (E) with respect to x is

$$-n^2 v(n, x', y, z) + n \frac{\pi - x'}{\pi} H_1(y, z) + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{n} F(x', y, z),$$

$$v(n, x', 0, z) = \frac{1}{n} H_3(x', z), \qquad v(n, x', \pi, z) = 0,$$

$$v(n, x', y, 0) = \frac{1}{n} H_5(x', y), \qquad v(n, x', y, \pi) = 0.$$

Next let $\bar{v}(n, n', z)$ be the sine transform of $v(n, x', y, z)$ with respect to x'. When $n' = n$, we get

$$-n^2 \bar{v}(n, y, z) + H_1(y, z) + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2} = \frac{1}{n} f(n, y, z),$$

$$\bar{v}(n, 0, z) = \frac{1}{n} h_3(n, z), \qquad \bar{v}(n, \pi, z) = 0, \tag{E'}$$

$$\bar{v}(n, y, 0) = \frac{1}{n} h_5(n, y), \qquad \bar{v}(n, y, \pi) = 0.$$

If we now multiply the equations in (E′) by the parameter n, we note that problems (D′) and (E′) are equivalent, and

$$u(n, y, z) = n\bar{v}(n, y, z). \tag{15}$$

Since $\bar{v}(n, y, z)$ is the iterated sine transform of $V(x, x', y, z)$, it follows, according formula (4), that

$$u(n, y, z) = n\, C\left\{\frac{1}{2} V^*(x, y, z)\right\}.$$

$$= -S\left\{\frac{1}{2} \frac{\partial}{\partial x} V^*(x, y, z)\right\}.$$

And therefore

$$U(x, y, z) = -\frac{1}{2} \frac{\partial}{\partial x} V^*(x, y, z),$$

which can be written (formula 3)

$$U(x, y, z) = \frac{1}{2} \frac{\partial}{\partial x} \int_{-\pi}^{\pi} V(x - x', x', y, z)\, dx'. \tag{16}$$

Formula (16) is the Duhamel integral extended from time to space coordinates.

Problem (D) can further be resolved into simpler problems. We first note that problem (E) can be written as

$$V(x, x', y, z) = \frac{\pi - x}{\pi} V_1(x', y, z) + V_2(x, x', y, z) + V_3(x, x', y, z),$$

where V_1, V_2 and V_3 are solutions of the problems:

$$\begin{aligned}
&\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} = 0,\\
&V_1(x', 0, z) = H_3(x', z), \qquad V_1(x', \pi, z) = 0,\\
&V_1(x', y, 0) = H_5(x', y), \qquad V_1(x', y, \pi) = 0;
\end{aligned} \tag{a}$$

$$\begin{aligned}
&\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_2}{\partial z^2} = 0,\\
&V_2(0, x', y, z) = \frac{\pi - x'}{\pi} H_1(y, z) - V_1(x', y, z)\\
&\qquad \equiv Q(x', y, z),\\
&V_2(\pi, x', y, z) = 0,\\
&V_2(x, x', 0, z) = V_2(x, x', \pi, z) = 0,\\
&V_2(x, x', y, 0) = V_2(x, x', y, \pi) = 0;
\end{aligned} \tag{b}$$

and

$$\begin{aligned}
&\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2} + \frac{\partial^2 V_3}{\partial z^2} = \frac{\pi - x}{\pi} F(x', y, z),\\
&V_3(0, x', y, z) = V_3(\pi, x', y, z) = 0,\\
&V_3(x, x', 0, z) = V_3(x, x', \pi, z) = 0,\\
&V_3(x, x', y, 0) = V_3(x, x', y, \pi) = 0.
\end{aligned} \tag{c}$$

It can be readily shown, see [3] p. 19-23, that the solution $U(x, y, z)$ of problem (D) then takes the following form:

$$U(x, y, z) = V_1(x, y, z) - \frac{1}{2}\frac{\partial}{\partial x} V_2^*(x, y, z) - \frac{1}{2}\frac{\partial}{\partial x} V_3^*(x, y, z).$$

The solution V_1 of the two dimensional problem (a) is the sum of solutions of the type of problem (B), Sec. 4, that is,

$$V_1(x', y, z) = \frac{1}{2} H_3(x', z)^* \frac{\partial}{\partial z} B_{01}(y, \pi - z)$$
$$+ \frac{1}{2} H_5(x', y)^* \frac{\partial}{\partial y} B_{01}(y, \pi - z)$$

We shall next proceed to obtain the solution V_2 of problem (b). Let $v_2(x, x', m, z)$ be the sine transform of $V_2(x, x', y, z)$ with respect to y and also let $\bar{v}_2(x, x', m, p)$ be

the iterated transform of $V_2(x, x', y, z)$ with respect to y and z. Then applying this iterated transformation to problem (b) we get

$$\frac{\partial^2 \bar{v}_2}{\partial x^2} - (m^2 + p^2)\bar{v}_2(x, x', m, p) = 0,$$
$$\bar{v}_2(0, x', m, p) = \bar{q}(x', m, p), \qquad \bar{v}_2(\pi, x', m, p) = 0, \tag{b'}$$

where $\bar{q}(x', m, p) = S\{q(x', m, z)\}$.

We shall now formulate the following basic problem (d) in three dimensions, and then express the solution V_2 in terms of its solution:

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} + \frac{\partial^2 U_0}{\partial z^2} = 0,$$
$$U_0(0, y, z) = \frac{\pi - y}{\pi} \cdot \frac{\pi - z}{\pi}, \qquad U_0(\pi, y, z) = 0, \tag{d}$$
$$U_0(x, 0, z) = U_0(x, \pi, z) = 0,$$
$$U_0(x, y, 0) = U_0(x, y, \pi) = 0.$$

Let $u_0(x, m, z)$ and $\bar{u}_0(x, m, p)$, respectively, be the sine transform of $U_0(x, y, z)$ with respect to y and its iterated transform with respect to y and z. Then

$$\frac{\partial^2 \bar{u}_0}{\partial x^2} - (m^2 + p^2)\bar{u}_0(x, m, p) = 0,$$
$$\bar{u}_0(0, m, p) = \frac{1}{mp}, \qquad \bar{u}_0(\pi, m, p) = 0. \tag{d'}$$

If we multiply the equations in problem (d′) by the function of the parameters $mp\bar{q}(x', m, p)$, it is evident in view of problem (b′) that

$$\bar{v}_2(x, x', m, p) = \bar{q}(x', m, p) mp\, \bar{u}_0(x, m, p).$$

That is

$$S\{v_2(x, x', m, z)\} = S\{q(x', m, z)\} mp S\{u_0(x, m, z)\}.$$

Since $u_0(x, m, 0) = u_0(x, m, \pi) = 0$, then $pS\{u_0(x, m, z)\} = C\{\partial u_0/\partial z\ (x, m, z)\}$. Hence the product of the transforms can be replaced by the sine transform of the convolution of the two functions. Upon applying the inverse transformation, we get

$$v_2(x, x', m, z) = \frac{1}{2}\, q(x', m, z) * m \frac{\partial}{\partial z} u_0(x, m, z)$$
$$= \frac{1}{2} \int_{-\pi}^{\pi} q(x', m, z - \lambda) m \frac{\partial}{\partial \lambda} u_0(x, m, \lambda)\, d\lambda,$$

which in turn can be written formally as follows:

$$S\{V_2(x, x', y, z)\} = \frac{1}{2} \int_{-\pi}^{\pi} S\{Q(x, y, z - \lambda)\} m\, S\left\{\frac{\partial}{\partial \lambda} U_0(x, y, \lambda)\right\} d\lambda.$$

Since again $\partial/\partial\lambda\, U_0(x, 0, \lambda) = \partial/\partial\lambda\, U_0(x, \pi, \lambda) = 0$, then, repeating the steps above, we obtain

$$V_2(x, x', y, z) = \frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(x, y - \mu, z - \lambda)\, \frac{\partial^2}{\partial\lambda\, \partial\mu}\, U_0(x, \mu, \lambda)\, d\lambda\, d\mu.$$

The solution $U_0(x, y, z)$ can be obtained with the aid of the Fourier sine series. The solution of problem (d′) is

$$\bar{u}_0(x, m, p) = \frac{\sinh (\pi - x)(m^2 + p^2)^{1/2}}{mp \sinh \pi\, (m^2 + p^2)^{1/2}}.$$

These functions are, except for the constant factor $\pi/2$, the Fourier sine coefficients for the function $u_0(x, m, z)$, which in turn are in like manner the sine coefficients for the function $U_0(x, y, z)$. Hence

$$U_0(x, y, z) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{\sinh (\pi - x)(m^2 + p^2)^{1/2}}{mp \sinh \pi(m^2 + p^2)^{1/2}} \sin pz \sin my.$$

Solution of problem (c). The iterated sine transform of (c) with respect to x and y is

$$\begin{aligned}&\frac{\partial^2 \bar{v}_3}{\partial x^2} - (n^2 + m^2)\bar{v}_3(n, x', m, z) = \frac{1}{n}\, f(x', m, z),\\ &\bar{v}_3(n, x', m, 0) = \bar{v}_3(n, x', m, \pi) = 0.\end{aligned} \qquad (c')$$

In terms of the Green's function $g(n, m, z, \mu)$ the solution of (c′) has the form

$$\bar{v}_3(n, x', m, z) = \int_0^{\pi} \frac{g(n, m, z, \mu) f(x', m, \mu)\, d\mu}{n}, \qquad (17)$$

where

$$g(n, m, z, \mu) = \frac{\sinh \mu(m^2 + n^2)^{1/2} \sinh (z - \pi)(m^2 + n^2)^{1/2}}{n\, m \sinh \pi(m^2 + n^2)^{1/2}}, \qquad \mu \leq z,$$

and with μ and z interchanged when $\mu \geq z$.

Performing the iterated inverse sine transformation with the aid of Fourier sine series, we get

$$V_3(x, x', y, z) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \bar{v}_3(n, x', m, z) \sin nx \sin my,$$

where $\bar{v}_3(n, x', m, z)$ is defined by (17).

GENERATION OF SURFACE WAVES BY A MOVING PARTITION*

BY

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Artificial waves in water are frequently generated by imparting rhythmic motion to a boundary of some sort. Two cases of wave production in deep water have been treated, for harmonic motion only, by Havelock.[1] The more interesting of these cases will be treated here by a different method, and expressions will be obtained for the initial phases as well as for the permanent regime; in addition, the solution will be extended to liquids of limited depth.

The harmonic case requires, for the most part, another two-dimensional solution of the Laplace equation, subject to a special set of boundary conditions. Besides the familiar condition at the free surface expressing the action of gravity, and perhaps a condition on a parallel bottom, the horizontal motion is here prescribed over a vertical boundary. The functions found are interesting in that they constitute an infinite variety of characteristic solutions, all asymptotic at infinity to the same functional form. This asymptotic form is the sine function that is familiar in the treatment of gravity waves and represents the only bounded solution when the free surface is unlimited in extent.

I. Case of Infinite Depth

Consider a semi-infinite body of incompressible, frictionless liquid having a free surface and of infinite depth. Let it be limited at the left by a boundary that executes infinitesimal displacements from an initial vertical plane position. Initially at time $t = 0$, let the liquid be at rest with a horizontal free surface.

Draw the y-axis downward in the initial plane of the vertical boundary and the x-axis perpendicular to this plane and in the horizontal plane that is occupied initially by the free surface. Then the motion excited by the motion of the boundary will be two-dimensional, occurring in planes parallel to the xy-plane, so that nothing is a function of z.

Let the infinitesimal horizontal displacement s of a point of the boundary at time t be

$$s = SF(y, t), \tag{1}$$

where S is a constant having the dimensions of length. Thus the function $F(y, t)$ is dimensionless. Initially, $F(y, 0) = 0$. It will be assumed that the function $F(y, t)$ is sufficiently smooth and vanishes sufficiently rapidly as $y \to \infty$ to justify the operations that are to be performed.

Since the liquid is assumed frictionless and incompressible, there will be a velocity potential satisfying Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

A solution is required for $x \geqq 0$, $y \geqq 0$, subject to two boundary conditions. Over the

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[1]Havelock, Phil. Mag. **8**, 569 (1929).

plane of the moving boundary the vertical component of velocity of the water is unrestricted, but the horizontal component must be the same as that of the boundary. Hence for $x = 0$,

$$v_x = -\frac{\partial \phi}{\partial x} = \frac{\partial s}{\partial t} = SF_t(y, t). \tag{2}$$

At the free surface the boundary condition for infinitesimal motions may be written[2] in terms of the elevation η of the surface:

$$\eta = \frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)_{y=0} \quad \text{(3a)}, \qquad \frac{\partial \eta}{\partial t} = \left(\frac{\partial \phi}{\partial y}\right)_{y=0}, \quad \text{(3b)}$$

where g is the acceleration due to gravity.

A solution of the differential equation satisfying these boundary conditions will be constructed in two steps. For the moment, imagine gravity to be absent. Then, since the displacement of the boundary remains small, the motion of the liquid can be regarded as due to a suitable distribution of horizontal line sources along the positive y-axis, continued above the free surface along negative y as negative image sources in order to keep the pressure constant on the free surface, or in the xz-plane. The velocity potential due to a line source located at $(0, \pm\beta)$ can be written $-A \log r^2 = -A \log [x^2 - (y \mp \beta)^2]$, where A is a constant and the flux toward one side is $2\pi A$; the desired flux from dy, however, is $v_x dy = SF_t(y, t)\, dy$ by (2). It is thus found that the desired velocity potential ϕ_1 and the corresponding horizontal velocity component v_{x1} are

$$\phi_1 = \frac{S}{2\pi}\int_0^\infty F_t(\beta, t) \log \frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2}\, d\beta, \tag{4}$$

$$v_{x1} = -\frac{\partial \phi_1}{\partial x} = \frac{S}{\pi}\int_0^\infty F_t(\beta, t)\left[\frac{x}{x^2 + (\beta - y)^2} - \frac{x}{x^2 + (\beta + y)^2}\right] d\beta,$$

where $F_t(\beta, t) = \partial F(\beta, t)/\partial t$.

It is readily seen that ϕ_1 is a solution of Laplace's equation and that $\partial\phi_1/\partial t = 0$ when $y = 0$. Consequently, the pressure remains zero at $y = 0$, as it must for infinitesimal motions in the absence of gravity. As $x \to 0$, the second term in the expression for v_{x1} vanishes. In the first term, however, the integrand is small except near $y = \beta$, so that in the limit as $x \to 0$, $F_t(\beta, t)$ may be replaced by $F_t(y, t)$ and the lower limit of the integral may be replaced by $-\infty$. Thus

$$\lim_{x \to 0} v_{x1} = \frac{S}{\pi} F_t(y, t) \int_{-\infty}^\infty \frac{x\, d\beta}{x^2 + (\beta - y)^2} = SF_t(y, t),$$

as is required by (2).

The corresponding upward velocity of the surface of the liquid is

$$v_{n1} = \left(\frac{\partial \phi_1}{\partial y}\right)_{y=0} = \frac{2S}{\pi}\int_0^\infty F_t(\beta, t)\, \frac{\beta\, d\beta}{x^2 + \beta^2}\,. \tag{5}$$

The effect of gravity may now be introduced by the following device. During each element of time dt the motion already assumed adds an increment to the elevation of

[2]Lamb, *Hydrodynamics*, Cambridge University Press; 6 ed., Sec. 227.

the surface represented by $v_{n1}\,dt$. In the absence of gravity these increments remain superposed upon each other without alteration. Under gravity, on the other hand, each increment undergoes transformation in the manner characteristic of an "initial elevation" of the same magnitude (the relevant theory is given in Lamb's *Hydrodynamics*,[3] y being there measured upward). The usual Fourier integral may be omitted here, however, since it may be verified by carrying out the integration in k that (5) can be written

$$v_{n1} = \left(\frac{\partial \phi_1}{\partial y}\right)_{y=0} = \frac{2S}{\pi} \int_0^\infty \cos kx\, dk \int_0^\infty F_t(\beta, t)\, e^{-k\beta}\, d\beta. \tag{6}$$

Now a standing wave with a surface elevation $\eta = \cos \sigma t \cos kx$, where $\sigma^2 = gk$, has an associated potential $g(\sin \sigma t/\sigma)e^{-ky} \cos kx$ (cf. Lamb, Sec. 238). Hence the potential for an oscillation reducing at time $t = t'$ to a surface elevation $v_{n1}\,dt'$ is, using (6),

$$\begin{gathered} d\phi = \frac{2}{\pi}\, g\, S\, dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t')\, e^{-ky} \cos kx\, Q(k, t')\, dk, \\ Q(k, t) = \int_0^\infty F_t(\beta, t)\, e^{-k\beta}\, d\beta. \end{gathered} \tag{7}$$

The total potential ϕ is then found by integrating (7) with respect to t', and adding ϕ_1 :

$$\phi = \phi_1 + \frac{2}{\pi}\, g\, S \int_0^t dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t')\, e^{-ky} \cos kx\, Q(k, t')\, dk. \tag{8}$$

The corresponding surface elevation is, from (3a), since $\partial\phi_1/\partial t = 0$ at $y = 0$,

$$\eta = \frac{2S}{\pi} \int_0^t dt' \int_0^\infty \cos \sigma(t - t') \cos kx\, Q(k, t')\, dk. \tag{9}$$

It is easily verified that ϕ as given by (8) satisfies Laplace's equation and the required boundary conditions (2) and (3b). At $x = 0$, $\partial\phi/\partial x = \partial\phi_1/\partial x$ and (2) is satisfied; in $\partial\eta/\partial t$, the term obtained from the upper limit equals $\partial\phi_1/\partial y$ as expressed in (6), and $gk = \sigma^2$.

These equations represent the motion caused by an arbitrary motion of the vertical boundary. In the harmonic case the expressions become simpler.

1. The harmonic case. As a special case, let the displacement of the vertical boundary after $t = 0$ be

$$s = SF(y, t) = Sf(y) \sin \omega t, \tag{10}$$

so that $F_t(y, t) = \omega f(y) \cos \omega t$. Then, replacing t' by $\tau = t - t'$ as the variable of integration, (8), (9) and (7) are replaced by

$$\begin{gathered} \phi = \phi_1 + \frac{2}{\pi}\, \omega g S \int_0^t d\tau \int_0^\infty (\cos \omega t \cos \omega\tau + \sin \omega t \sin \omega\tau) \\ \times\, \sigma^{-1} \sin \sigma\tau\, e^{-ky} \cos kx\, K(k)\, dk, \end{gathered} \tag{11}$$

[3] Lamb, *Hydrodynamics*, Cambridge University Press, 6 ed., Sec. 238.

$$\eta = \frac{2}{\pi} \omega S \int_0^t d\tau \int_0^\infty (\cos \omega t \cos \omega \tau + \sin \omega t \sin \omega \tau) \times \cos \sigma \tau \cos kx \, K(k) \, dk, \tag{12}$$

$$K(k) = \int_0^\infty e^{-k\beta} f(\beta) \, d\beta. \tag{13}$$

These expressions represent a non-harmonic motion of the water, caused by a harmonic motion of the boundary after an initial state of complete rest. As $t \to \infty$, however, the integrals representing the coefficients of $\sin \omega t$ and $\cos \omega t$ will approach fixed limits, provided $f(y)$ satisfies the usual restrictions. Thus, the entire motion becomes more and more nearly harmonic.

There must exist, therefore, an ideal solution representing exactly harmonic motion, in which (10) holds at all times. To obtain the corresponding amplitude functions in ϕ, which may be written

$$\phi = \phi_1 + \psi_1(x, y) \cos \omega t + \psi_2(x, y) \sin \omega t,$$

we carry out the integrations in τ and take limits as $t \to \infty$. The amplitude functions thus found are:

$$\psi_1(x, y) = \lim_{t\to\infty} \frac{\omega g S}{\pi} \int_0^\infty \left(\frac{1 - \cos(\sigma + \omega)t}{\sigma + \omega} + \frac{1 - \cos(\sigma - \omega)t}{\sigma - \omega} \right) G \, dk + \frac{\phi_1}{\cos \omega t},$$

$$\psi_2(x, y) = \lim_{t\to\infty} \frac{\omega g S}{\pi} \int_0^\infty \left(-\frac{\sin(\sigma + \omega)t}{\sigma + \omega} + \frac{\sin(\sigma - \omega)t}{\sigma - \omega} \right) G \, dk,$$

$$G = \sigma^{-1} e^{-ky} \cos kx \, K(k).$$

The values of the limits are obtained at once from the formulas:

$$\lim_{t\to\infty} \int_a^b f(x) \frac{\sin tx}{x} \, dx = \pi f(0) \quad \text{or} \quad 0 \tag{14}$$

according as either $a < 0 < b$ or a and b have the same sign;

$$\lim_{t\to\infty} \int_{-a}^a f(x) \frac{1 - \cos tx}{x} \, dx = \int_{-a}^a f(x) \frac{dx}{x}, \tag{15}$$

where the principal value of the last integral is taken.

These formulas can be reduced to the familiar one, proved in many books:[4]

$$\lim_{t\to\infty} \int_a^b f(x) \sin tx \, dx = 0.$$

In (15), for example, the left-hand member can be written, because of the symmetry of $(1 - \cos tx)$

$$\lim_{t\to\infty} \frac{1}{2} \int_{-a}^a \frac{f(x) - f(-x)}{x} (1 - \cos tx) \, dx.$$

[4]Whittaker and Watson, *Modern Analysis*, Cambridge University Press, 3rd ed., Sec. 9.41.

With mild restrictions upon $f(x)$, the first fraction here will be finite at $x = 0$, and the $\cos tx$ term will give zero in the limit; for the remainder, note that

$$\int_{-a}^{a} f(x) \frac{dx}{x} = - \int_{-a}^{a} f(-x) \frac{dx}{x} .$$

The integral in formula (14) when $a < 0 < b$ can be reduced in an analogous manner to the integral,

$$\int_{-\infty}^{\infty} \sin tx \, dx/x = \pi \qquad \text{for} \qquad t > 0.$$

Using these formulas, the cosine terms in ψ_1 and the $(\sigma + \omega)$ term in ψ_2 are zero, whereas the $(\sigma - \omega)$ term in ψ_2 can be evaluated after changing from dk to $d(\sigma - \omega)$; since $\sigma^2 = gk$, $dk = 2\sigma \, d\sigma/g$. Thus (11) becomes, using (4) and (10),

$$\phi = 2\omega S\left[\frac{1}{4\pi} \int_0^{\infty} f(\beta) \log \frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2} \, d\beta \right.$$
$$\left. + \frac{1}{\pi} \int_0^{\infty} \frac{dk}{k - k_0} e^{-ky} \cos kx \, K(k) \, dk\right] \cos \omega t \tag{16}$$
$$+ 2\omega S \exp(-k_0 y) \cos k_0 x \, K(k_0) \sin \omega t,$$

where

$$k_0 = \omega^2/g. \tag{17}$$

Nearly the same integrals occur in (12), which becomes

$$\eta = \frac{2\omega^2 S}{g}\left[\cos k_0 x \, K(k_0) \cos \omega t - \frac{1}{\pi} \sin \omega t \int_0^{\infty} \frac{dk}{k - k_0} \cos kx \, K(k) \, dk\right]. \tag{18}$$

The boundary conditions (2) and (3b) are again easily verified. In $\partial\phi/\partial y$, $-k/(k - k_0) = -1 - k_0/(k - k_0) = -1 - \omega^2/g(k - k_0)$; the first term serves to cancel the contribution from the first integral when $y = 0$ because of (4) and (6).

Equations (16) and (18) represent exactly harmonic motion, but they may also be used at a given point as approximate expressions for the motion after a start from rest, provided sufficient time has elapsed. It is necessary that, as σ varies past the value ω in the integrations in (11) and (12), $\sin \sigma t$ or $\cos \sigma t$ shall execute many periods while $\cos kx$ varies little from $\cos k_0 x$. Consequently, a good approximation to the limits as $t \to \infty$ will be obtained. This requires that

$$\omega t \gg k_0 x = \omega^2 x/g = 2\pi x/\lambda$$

where $\lambda = 2\pi g/\omega^2$ represents the wave length of small traveling waves of frequency $\omega/2\pi$. In other words, the total number of waves emitted from the boundary since the start, as represented by $\omega t/2\pi$, must greatly exceed x/λ or the number of waves included between the boundary and the point x.

On the other hand, as x increases beyond the first few wave lengths from the boundary, the exactly harmonic motion approximates more and more closely simple traveling waves having the frequency of the boundary. We can show this by transforming the remaining integrals in k.

Putting $\zeta = k + im$ and integrating around the first quadrant of the ζ-plane, indented at $x = k_0$, we find that

$$\oint \frac{\exp(-\gamma\zeta + i\alpha\zeta)}{\zeta - k_0}\, d\zeta = \int_0^\infty \frac{\exp(-\gamma k + i\alpha k)}{k - k_0}\, dk$$

$$- i\pi \exp(-\gamma k_0 + i\alpha k_0) - i \int_0^\infty \frac{\exp(-i\gamma m - \alpha m)}{im - k_0}\, dm = 0$$

for $\gamma > 0$ and $\alpha > 0$; the principal value of the k integral is intended. The last integral can also be written

$$- \int_0^\infty \frac{im + k_0}{m^2 + k_0^2} \exp(-i\gamma m - \alpha m)\, dm.$$

Hence, taking real parts, we obtain the formula,

$$\int_0^\infty \frac{dk}{k - k_0} e^{-\gamma k} \cos \alpha k\, dk = -\pi \exp(-\gamma k_0) \sin \alpha k_0$$

$$+ \int_0^\infty \frac{m \cos \gamma m - k_0 \sin \gamma m}{m^2 + k_0^2} e^{-\alpha m}\, dm.$$

For our present purpose, change m to k, set $\alpha = x$ and $\gamma = \beta + y$ in ϕ or $\gamma = \beta$ in η; the factor $e^{-\beta k}$ is to be found in the integral for K, as given in (13). It is then found from (16), (18), (13), that

$$\begin{aligned}\phi = {} & 2\omega S\, K(k_0) \exp(-k_0 y) \sin(\omega t - k_0 x) \\ & + \frac{2}{\pi} \omega S \cos \omega t \int_0^\infty f(\beta)\, d\beta \left[\frac{1}{4} \log \frac{x^2 + (\beta + y)^2}{x^2 + (\beta - y)^2} \right. \\ & \left. + \int_0^\infty \frac{k \cos k(\beta + y) - k_0 \sin k(\beta + y)}{k^2 + k_0^2} e^{-kx}\, dk \right],\end{aligned} \tag{19}$$

$$\begin{aligned}\eta = {} & \frac{2\omega^2 S}{g} \left[K(k_0) \cos(\omega t - k_0 x) \right. \\ & \left. - \frac{1}{\pi} \sin \omega t \int_0^\infty f(\beta)\, d\beta \int_0^\infty \frac{k \cos \beta k - k_0 \sin \beta k}{k^2 + k_0^2} e^{-kx}\, dk \right].\end{aligned} \tag{20}$$

The first term in (19) or (20) represents a train of harmonic waves traveling outward from the moving vertical boundary. In terms of the wave length, $\lambda = 2\pi/k_0 = 2\pi g/\omega^2$, the surface amplitude of these waves is, from (20) and (13),

$$A = \frac{4\pi S}{\lambda} \int_0^\infty \exp(-2\pi y/\lambda) f(y)\, dy. \tag{21}$$

The second term in (19) or (20) represents a local disturbance near the wave-making boundary, superposed upon the waves in time quadrature with them. The resultant disturbance near the boundary in the exactly harmonic motion will thus have an amplitude in general larger than that of the distant waves.

At the boundary itself, the expression for the surface amplitude in the exactly harmonic motion becomes simpler. With $x = 0$, in terms of $k' = k - k_0$, the integral in (18) becomes, using (13),

$$\int_0^\infty f(\beta)\, d\beta \int_{-k_0}^\infty \frac{dk'}{k'} \exp\,[-\beta(k_0 + k')] = -\int_0^\infty \exp\,(-\beta k_0) Ei(\beta k_0) f(\beta)\, d\beta,$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t}\, dt = -\int_{-x}^\infty \frac{e^{-t}}{t}\, dt.$$

Hence at $x = 0$, in terms of $\lambda = 2\pi g/\omega^2$, from (18) and (13),

$$\eta = \frac{4\pi S}{\lambda} \int_0^\infty \left[\cos \omega t + \frac{1}{\pi} Ei(k_0 y) \sin \omega t\right] \exp\,(-k_0 y) f(y)\, dy. \tag{22}$$

Here, as $y \to 0$, $Ei(k_0 y)$ becomes logarithmically infinite; nevertheless, if $f(0+)$ is absolutely bounded and integrable, the integral will converge.

The pressure on the boundary is also obtainable in closed form, for the case of exactly harmonic motion. It is $p = \rho(\partial\phi/\partial t)_{x=0}$. For the evaluation of the integral in (19), the following formula is easily inferred from formulas given in Bierens de Haan's table of integrals:

$$\int_0^\infty \frac{k \cos pk - q \sin pk}{k^2 + q^2}\, dk = -e^{-pq} Ei(pq),$$

where p and q are constants. For the present case, take $q = k_0$, $p = \beta + y$. The pressure at depth y is thus found to be, using (13),

$$\begin{aligned} p = {} & 2\omega^2 \rho S \cos \omega t \exp\,(-k_0 y) \int_0^\infty \exp\,(-k_0\beta) f(\beta)\, d\beta \\ & - \frac{2\omega^2 \rho S}{\pi} \sin \omega t \int_0^\infty f(\beta) \left\{ \frac{1}{4} \log \frac{(\beta + y)^2}{(\beta - y)^2} \right. \\ & \left. - \exp\,[-k_0(\beta + y)] Ei[k_0(\beta + y)] \right\} d\beta. \end{aligned} \tag{23}$$

The second term in p is a quadrature component that, on the whole, does no work on the boundary; the first term provides the energy carried away by the waves.

Any motion consistent with the assumed harmonic boundary conditions can be resolved into the motion already described and a complementary component satisfying the condition of zero horizontal velocity over the vertical plane $x = 0$.

2. The pure-wave case. If $f(y) = \exp\,(-k_0 y)$, the boundary moves as does a vertical layer of fluid particles during the passage of harmonic waves; the entire motion must then reduce to that of the wave train.

Now, if $f(y) = \exp\,\{-2\pi y/\lambda\}$, (21) gives at once the amplitude $A = S$, which is correct, since in deep waves the vertical and horizontal amplitudes are equal. That the local disturbance vanishes completely in this case is most easily seen if the corresponding term in ϕ is put into Havelock's form.

The first of the following equations is easily obtained, and integration of it with respect to α yields the second:

$$\int_0^\infty e^{-\alpha x}(\cos px - \cos qx)\, dx = \frac{\alpha}{\alpha^2 + p^2} - \frac{\alpha}{\alpha^2 + q^2};$$

$$\int_0^\infty \frac{\cos px - \cos qx}{x} e^{-\alpha x}\, dx = \frac{1}{2}\log\frac{\alpha^2 + q^2}{\alpha^2 + p^2}.$$

The constant of integration in the latter formula must be zero since both members of the equation vanish as $\alpha \to \infty$.

Using this last formula with $x = k$, $\alpha = x$, $p = \beta - y$ and $q = \beta + y$ to transform the logarithm in (19), we obtain as the coefficient of $f(\beta)d\beta$ in (19),

$$\int_0^\infty \left[\frac{1}{2}\,\frac{\cos(\beta - y)k - \cos(\beta + y)k}{k} + \frac{k\cos k(\beta + y) - k_0 \sin k(\beta + y)}{k^2 + k_0^2}\right] e^{-kx}\, dk,$$

and, after consolidating, the integral in β in (19) becomes

$$\int_0^\infty f(\beta)\, d\beta \int_0^\infty (k\cos\beta k - k_0 \sin\beta k)\,\frac{k\cos ky - k_0\sin ky}{k(k^2 + k_0^2)} e^{-kx}\, dk.$$

The entire expression for ϕ in (19) then agrees with Havelock's expression except for a change of notation. Furthermore, it is easily verified, by carrying out the β integration, that if $f(y) = \exp(-k_0 y)$, the expression just written vanishes. The right-hand member of (19) thus reduces in this case to the first term, representing the waves. In the same way the $\sin \omega t$ term in (20) disappears.

II. Case of Finite Depth

Let it now be assumed that the liquid has only a finite depth h. Nothing essentially new is introduced by this limitation. The principal formulas will accordingly be written down with a minimum of explanation.

In using the source method, infinite trains of line images are now required in order to preserve the boundary condition both on the bottom and on the free surface. The sources associated with an element dy of the boundary at a depth $y = \beta$ fall into two series, beginning respectively at $y = \pm\beta$; in each series the spacing is $2h$ and the signs alternate, since reflection in the free surface reverses the sign whereas reflection in the bottom does not. This alternation of sign suggests use of the complex potential $C \log \tan az$, whose real part, if C and a are real, is $(C/2)$ [log (cosh $2ay$ − cos $2ax$) − log (cosh $2ay$ + cos $2ax$)]. The proper periodicity is obtained if $a = \pi/4h$; and for small x this expression reduces, except for an added constant, to $(C/2)\log(x^2 + y^2)$ or to the potential of a line source at the origin. For one train of sources, y is replaced by $(y - \beta)$, for the other, by $(y + \beta)$. It is thus found that, for the motion in the absence of gravity, (2) and (3) representing the potential and the corresponding upward surface velocity are replaced by

$$\phi_1' = \frac{S}{2\pi}\int_0^h F_t(\beta, t)\log\left(\frac{\cosh bx - \cos b(\beta + y)}{\cosh bx + \cos b(\beta + y)}\,\frac{\cosh bx + \cos b(\beta - y)}{\cosh bx - \cos b(\beta - y)}\right) d\beta, \quad (24)$$

$$v_{n1}' = \left(\frac{\partial\phi_1'}{\partial y}\right)_{y=0} = \frac{S}{h}\cosh bx \int_0^h \frac{F_t(\beta, t)\sin b\beta}{\sinh^2 bx + \sin^2 b\beta}\, d\beta, \quad (25)$$

with

$$b = \frac{\pi}{2h}.$$

The expression for v'_{n1} is then replaced by a Fourier integral:

$$v'_{n1}(x, t) = \frac{2}{\pi} \int_0^\infty \cos kx \, dk \int_0^\infty v'_{n1}(\alpha, t) \cos k\alpha \, d\alpha. \tag{26}$$

The formula needed to reduce this integral can be obtained by integrating $e^{ikz} \cosh bz(\sinh^2 bz + \sin^2 b\beta)^{-1} \, dz$ around the upper half plane for $z = x + iy$. Poles occur on the imaginary axis where $\sin by = \pm \sin b\beta$, or $y = \beta + n\pi/b$, $y = -\beta + (n + 1)\pi/b$, where $n = 0, 1, 2, \cdots$. The sum of the residues thus leads to a series, namely, $1 - \exp\{-k\pi/b\} + \exp\{-2k\pi/b\} \cdots = (1 - \exp\{-k\pi/b\})^{-1}$. The real part of the integral is thus found to yield the formula

$$\int_0^\infty \frac{\cos kx \cosh bx}{\sinh^2 bx + \sin^2 b\beta} \, dx = \frac{\pi}{2\, b \sin b\beta} \left[e^{-k\beta} + \frac{2 \sinh k\beta}{e^{\pi k/b} + 1} \right]. \tag{27}$$

By means of this formula, (26) with $v'_{n1}(\alpha, t)$ inserted from (25) reduces to

$$v'_{n1} = \frac{2S}{\pi} \int_0^\infty \cos kx \, Q'(k, t) \, dk, \qquad Q'(k, t) = \int_0^h F_t(\beta, t) \left(e^{-k\beta} + \frac{2 \sinh k\beta}{e^{2kh} + 1} \right) d\beta.$$

This equation for v'_{n1} replaces (6). With the usual modifications to fit the finite depth, the former procedure then gives for the total velocity potential ϕ' and surface elevation η' in place of (8) and (9),

$$\phi' = \phi'_1 + \frac{2gS}{\pi} \int_0^t dt' \int_0^\infty \sigma^{-1} \sin \sigma(t - t') \frac{\cosh k(h - y)}{\cosh kh} \cos kx \, Q'(k, t') \, dk, \tag{28}$$

$$\eta' = \frac{2S}{\pi} \int_0^t dt' \int_0^\infty \cos \sigma(t - t') \cos kx \, Q'(k, t') \, dk, \tag{29}$$

where now

$$\sigma^2 = gk \tanh kh.$$

The boundary conditions are easily verified, including the new one that $\partial\phi/\partial y = 0$ when $y = h$.

3. The harmonic case. The treatment of this case then proceeds as before. It is found that, in the exactly harmonic case where the displacement of the boundary is at all times $s = Sf(y) \cos \omega t$, instead of (16) and (18),

$$\phi' = \phi'_1 + 2\omega S \left\{ \frac{\cosh k_1(h - y)}{\cosh k_1 h} \frac{K'(k_1)}{\tanh k_1 h + k_1 h \operatorname{sech}^2 k_1 h} \sin \omega t \cos k_1 x \right.$$
$$\left. + \frac{\cos \omega t}{\pi} \int_0^\infty \frac{\cosh k(h - y)}{\cosh kh} \cos kx \frac{K'(k) \, dk}{k \tanh kh - k_1 \tanh k_1 h} \right\} \tag{30}$$

where k_1 is the positive real root of the equation

$$\omega^2 = gk_1 \tanh k_1 h \tag{31}$$

and

$$K'(k) = \int_0^h f(\beta)\left[e^{-k\beta} + \frac{2\sinh k\beta}{e^{2kh}+1}\right] d\beta, \tag{32}$$

$$\begin{aligned}\eta' = \frac{2\omega^2 S}{g}\Bigg\{&\cos\omega t\cos k_1 x\,\frac{K'(k_1)}{\tanh k_1h + k_1h\,\mathrm{sech}^2\,k_1h} \\ &- \frac{\sin\omega t}{\pi}\int_0^\infty \cos kx\,\frac{K'(k)\,dk}{k\tanh kh - k_1\tanh k_1h}\Bigg\}.\end{aligned} \tag{33}$$

In deducing (30) from (28), an equation is first obtained that is identical with (11) except that ϕ and ϕ_1 are replaced by ϕ' and ϕ_1', $K(k)$ by $K'(k)$ as given in (32) above, and e^{-ky} by $\cosh k(h - y)/\cosh kh$. With the same changes, the expressions given under Eq. (11) for $\psi_1(x, y)$ and $\psi_2(x, y)$ become the amplitude functions ψ_1' and ψ_2' for ϕ', such that $\phi' = \phi_1' + \psi_1'(x, y)\cos\omega t + \psi_2'(x, y)\sin\omega t$. In the further reduction, however, k_1 replaces k_0, and the relation $\sigma^2 - \omega^2 = g(k - k_0)$ is replaced by $\sigma^2 - \omega^2 = g(k\tanh kh - k_1\tanh k_1h)$, so that the latter expression in parentheses replaces $(k - k_0)$; furthermore, the relation $dk = 2\sigma\,d\sigma/g$ is replaced by $dk = 2\sigma\,d\sigma/g(\tanh kh + kh\,\mathrm{sech}^2\,kh)$, so that at $k = k_1$ an additional factor $(\tanh k_1h + k_1h\,\mathrm{sech}^2\,k_1h)$ is introduced. The differences between (30) and (16) are thus explained. The expression for η' is then easily obtained from (3a).

The waves at large x can be discovered as before by transforming the remaining integrals by contour integration. Here, the expression $k\tanh kh - k_1\tanh k_1h$ appears in (30) and (33) where $(k - k_0)$ appears in (16) and (18); hence at the pole, which occurs here at $k = k_1$, the following additional factor is obtained in the denominator:

$$\left[\frac{d}{dk}(k\tanh kh - k_1\tanh k_1h)\right]_{k=k_1} = \tanh k_1h + k_1h\,\mathrm{sech}^2\,k_1h.$$

The terms in $\sin\omega t$ thus combine again with the $\cos\omega t$ terms to represent traveling waves at large x. The remaining expressions are complicated, but they involve x only in a factor of the form $\exp(-k'x)$ where k' is real and positive, and so represent again a local disturbance near the boundary.

The surface amplitude A of the waves can be inferred from (33) with (31) and (32):

$$\begin{aligned}A = \frac{4\pi S}{\lambda_1}&(1 + k_1h\,\mathrm{sech}\,k_1h\,\mathrm{csch}\,k_1h)^{-1} \\ &\times\int_0^h f(y)\left[\exp(-k_1y) + \frac{2\sinh k_1y}{\exp(2k_1h) + 1}\right]dy\end{aligned} \tag{34}$$

where $k_1 = 2\pi/\lambda_1$ in terms of the wave length λ_1. As $h \to \infty$, this expression reverts to that for deep water as given in (21).

If, on the other hand, h/λ_1 is small, so is k_1y throughout the range of integration, and $k_1h\,\mathrm{sech}\,k_1h\,\mathrm{csch}\,k_1h = 1$, nearly; thus, approximately,

$$A = \frac{2\pi S}{\lambda_1}\int_0^h f(y)\,dy. \tag{35}$$

This is easily seen to be the correct expression for canal waves.

ON STEADY, LAMINAR TWO-DIMENSIONAL JETS IN COMPRESSIBLE VISCOUS GASES FAR BEHIND THE SLIT*

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1. Introduction. Schlichting and Bickley[1] solved the problem of a laminar, steady, two-dimensional jet in an incompressible viscous fluid flowing through a narrow slit in a wall and then mixing with the surrounding medium at rest. In the present paper the problem of such a jet in a compressible viscous fluid is solved. The following equations are taken into account: equations of motion, continuity, energy and state. The coefficients of viscosity and thermal conductivity are assumed to be functions of temperature. In order not to obscure the problem by many items of a simple algebraic nature this paper presents only an outline of the method of attack. The method enables one to find the distribution of the velocity, density etc. far behind the slit.

2. Basic equations. Assuming that the coefficients of viscosity μ and heat conductivity K are variable functions,† one obtains the following equations (equation of motion, continuity, state and energy):

$$\begin{aligned} \rho[\mathbf{V}_t + \operatorname{grad}(\tfrac{1}{2}\mathbf{V}^2) - (\mathbf{V}\times\boldsymbol{\omega})] &= \rho\mathbf{F} - \operatorname{grad} p \\ &- \tfrac{2}{3}(\operatorname{grad}\mu)(\operatorname{div}\mathbf{V}) + 2(\operatorname{grad}\mu\cdot\nabla)\mathbf{V} + (\operatorname{grad}\mu)\times\boldsymbol{\omega} \\ &+ \tfrac{1}{3}\mu\operatorname{grad}(\operatorname{div}\mathbf{V}) + \mu\nabla^2\mathbf{V}, \end{aligned} \tag{1}$$

$$\rho_t + \nabla(\rho\mathbf{V}) = 0, \qquad p = R\rho T, \tag{2}$$

$$\begin{aligned} Jc_v\rho(T_t + \mathbf{V}\cdot\operatorname{grad} T) + p\operatorname{div}\mathbf{V} \\ = J[(\operatorname{grad} K)\cdot(\operatorname{grad} T) + K\operatorname{div}(\operatorname{grad} T)] + \phi, \end{aligned} \tag{3}$$

$$\begin{aligned} \phi = \mu\{2\nabla[(\mathbf{V}\cdot\nabla)\mathbf{V}] + \omega^2 - 2\mathbf{V}\cdot\operatorname{grad}(\operatorname{div}\mathbf{V}) \\ - \tfrac{2}{3}(\operatorname{div}\mathbf{V})^2\}, \end{aligned} \tag{3a}$$

where $\quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{V}$.

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[1]H. Schlichting, *Laminare Strahlausbreitung*, Z. angew. Math. Mech., **13**, 261-263 (1933). W. G. Bickley, *The plane jet*, The London, Edinburgh and Dublin Phil. Mag. (7) **23**, 727-731 (1937). Also, *Modern developments in fluid dynamics*, ed. by S. Goldstein, Clarendon Press, Oxford, 1938. Bickley obtained the solution in the form of hyperbolic functions. Schlichting connected a series in ascending powers with an asymptotic solution, and obtained different numerical results.

†It is quite enough to assume that both μ and K are functions of the temperature only and not of the pressure. Both functions may be assumed to be polynomials or power series in T with properly selected coefficients (convergent series).

Assume steady two-dimensional flow in rectangular coordinates without external forces, with the z-axis in the vertical direction and with the equation of state included in the equation of energy. The equations may be presented in the form:

$$\begin{aligned} \rho u u_x - \mu u_{yy} &= -p_x - \rho u_y v + \mu_y u_y + 1/3\, \mu v_{xy} \\ &\quad + 4/3(\mu u_x)_x - 2/3\, \mu_x v_y + \mu_y v_x\,, \end{aligned} \tag{4}$$

$$\begin{aligned} \rho u v_x - 4/3\, \mu v_{yy} &= -p_y - \rho v v_y + 4/3\, \mu_y v_y + 1/3\, \mu u_{xy} \\ &\quad + \mu_x u_y - 2/3\, \mu_y u_x + (\mu v_x)_x\,, \end{aligned} \tag{5}$$

$$u\rho_x = -\rho(u_x + v_y) - \rho_y v, \tag{6}$$

$$\begin{aligned} J(c_v \rho u T_x - K T_{yy}) &= -R\rho T(u_x + v_y) - J c_v \rho v T_y \\ &\quad + J[(KT_x)_x + K_y T_y] + \phi, \end{aligned} \tag{7}$$

$$\phi = \mu\{4/3[u_x^2 - u_x v_y + v_y^2] + (u_y + v_x)^2\}, \tag{7a}$$

where subscripts denote partial differentiation.

A further assumption is that the jet cross sections taken into consideration are located very far downstream behind the slit. In an application of a series expansion in negative powers of x this assumption enables one to neglect consistently the higher powers of $1/x$. Consequently the expansion is an asymptotic one for large x.*

A brief explanation of the method of solution of the set of differential equations will be given. The coefficients on the left-hand sides of those equations will be assumed so as to obtain ordinary linear differential equations of the second order. On the right-hand sides the results from the preceding approximations will be used. The results will be represented in series:**

$$u = \sum_1^n u_i\,, \qquad v = \sum_1^n v_i\,, \qquad \rho = \rho_\infty + \sum_1^n \rho_i\,, \qquad T = T_\infty + \sum_1^n T_i\,.$$

3. Initial approximation. Assume that the gas-jet flows through a narrow slit in the wall of height h. The velocity distribution across the cross section of the slit being constant, one may write: $M = h\rho_0 u_0^2$, where M denotes the rate at which the momentum flows across the initial section of the jet. This relation permits one to find the value u_0. The calculations given below will present the velocity, density, and temperature patterns across the jet at some distance from the slit. The value u_0 will serve as an auxiliary parameter.

4. First approximation.

(a) *Longitudinal velocity component.* Put $u_1 \sim x^{-p} f(\eta)$, $\eta \sim y x^{-q}$. Since the rate M must be constant in all the cross sections, one has, in the first approximation,

*The applied method is, strictly speaking, an extension of Goldstein's method in incompressible flow. See: S. Goldstein, "On the Two-Dimensional Steady Flow of a Viscous Fluid Behind a Solid Body," Proc. Roy. Soc. of London, A, **142** (1933), p. 545-562.

**Throughout the paper the subscript zero denotes the values referring to the conditions inside the container from which the gas flows out. The subscript ∞ refers to the conditions in the undisturbed gas at rest outside the container.

$$M = 2\rho_\infty \int_0^\infty u_1^2\, dy \sim \int_0^\infty x^{-2p} x^q f^2(\eta)\, d\eta.$$

Hence $2p = q$. Another condition is that uu_{1x} must be of the same degree in x as u_{1yy}, or $p + 1 = p + 2q$. These two conditions give the values: $p = 1/4$, $q = 1/2$.

The longitudinal velocity component will always be calculated from the first equation of motion. In the first approximation put:* $v = 0$, $\mu = \mu_\infty$, $\rho = \rho_\infty$, $uu_x \approx u_0 u_{1x}$ and neglect the variation of the pressure, i.e., $p_x = 0$. In this approximation only the terms of degree $x^{-5/4}$ will be retained. One easily obtains

$$\rho_\infty u_0 u_{1x} - \mu_\infty u_{1yy} = 0. \tag{8}$$

Introduce a non-dimensional coordinate η:

$$\eta = y\left(\frac{u_0}{4\nu_\infty x}\right)^{1/2}, \qquad \nu_\infty = \frac{\mu_\infty}{\rho_\infty}; \qquad \eta_x = -\frac{1}{2}x^{-1}\eta, \qquad \eta_y = \left(\frac{u_0}{4\nu_\infty x}\right)^{1/2};$$

$$u_1 = A_1 u_0 x^{-1/4} f_1(\eta). \tag{9}$$

The boundary conditions are that u_1 must tend to zero when $x \to \infty$, that u_1 must tend exponentially to zero when $y \to \infty$ and from the symmetry conditions that u_1 must be an even function of y, i.e., $u_{1y} = u_{1\eta} = 0$ for $y = \eta = 0$. After some elementary transformation one obtains

$$f_1'' + 2\eta f_1' + f_1 = 0. \tag{10}$$

Put

$$f_1 = k_1 \exp(-\eta^2)$$

and obtain

$$k_1'' - 2\eta k_1' - k_1 = 0. \tag{11}$$

This ordinary linear differential equation of the second order has a solution in the form of a series (see Appendix). The final form is

$$u_1 = A_1 u_0 x^{-1/4} k_1 \exp(-\eta^2), \tag{12}$$

with all the boundary conditions fulfilled.** The value of A_1 will be calculated from the condition that the rate M must be constant in all the cross sections. Retaining from k_1 only one term gives:

$$M = 4A_1^2 u_0^{3/2} \rho_\infty \nu_\infty^{1/2}\left(\frac{\pi}{2^3}\right)^{1/2} \qquad \text{with} \qquad \int_0^\infty \exp(-2\eta^2)\, d\eta = \left(\frac{\pi}{2^3}\right)^{1/2}, \tag{13}$$

$$A_1 = \pm M^{1/2} u_0^{-1}\left(\frac{u_0}{2\pi\mu_\infty\rho_\infty}\right)^{1/4} \text{ in.}^{1/4}\ {*}{*}{*}\ . \tag{14}$$

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*See the explanation below for the selection of ρ_∞ instead of ρ_0.

**Another solution may be obtained by putting

$$f_1 = k_{12} \exp[-\eta^2/2], \qquad k_{12}'' - \eta^2 k_{12} = 0 \text{ (see appendix)}.$$

***Taking into account the second term will give the result:

$$\left(\frac{1}{4}\right)\left[\int_0^\eta \exp(-\eta^2)\, d\eta - \eta \exp(-\eta^2)\right].$$

This expression will add 25% to the value of M.

(b) *Transverse velocity component.* This component will always be calculated from the second equation of motion. But in order to find the highest degree in x of v, one calculates the value of v_{t1} from the continuity equation for the incompressible fluid: $u_{1x} + v_{t1y} = 0$, and obtains

$$v_{t1} = A_2 u_0 x^{-3/4} \int_0^\eta [(1 - 4\eta^2)k_1 + 2\eta k_1'] \exp(-\eta^2)\, d\eta + C. \tag{15}$$

The constant of integration C must be equal to zero to fulfill the boundary conditions. The component v must satisfy the following conditions: it must tend to zero when $x \to \infty$, it must tend exponentially to zero when $y \to \infty$, and from the symmetry conditions it must be an odd function of y or η, i.e., $v = 0$ for $y = \eta = 0$. The final result is:

$$\begin{aligned} v_{t1} &= A_2 u_0 x^{-3/4}[2\eta k_1 \exp(-\eta^2) - g_{t1}], \\ g_{t1} &= \int_0^\eta k_1 \exp(-\eta^2)\, d\eta, \qquad A_2 = \frac{1}{2} A_1 \left(\frac{\nu_\infty}{u_0}\right)^{1/2} \text{in.}^{3/4}. \end{aligned} \tag{16}$$

Although not all the boundary conditions are fulfilled, it is obvious that the value found for v may be considered as the third approximation to v $(\sim x^{-3/4})$. Hence $v_1 = 0$.*

(c) *Density.* The density will always be calculated from the continuity equation. It must be equal to ρ_∞ when $x \to \infty$ and $y \to \infty$. Hence let us put $\rho = \rho_\infty + \sum_1^\infty \rho_i$. On the left-hand side of the equation of continuity put the values $u \approx u_0$, $\rho = \rho_\infty + \rho_1$, and on the right-hand side the values u_1, v_1, and $\rho = \rho_0$. Retaining the terms of degree $x^{-5/4}$ one obtains: $u_0 \rho_{1x} = -\rho_0 u_{1x}$, or

$$\rho_1 = -\rho_0 u_0^{-1} u_1 + C = -A_1 \rho_0 x^{-1/4} k_1 \exp(-\eta^2), \qquad C = 0.^{**} \tag{17}$$

The boundary conditions for ρ_1 are identical with those for u_1.

(d) *Temperature.* The temperature will always be calculated from the energy equation. In the first approximation only the terms of degree $x^{-5/4}$ will be retained. The temperature at infinity must be equal to T_∞. Hence put $T = T_\infty + \sum_1^\infty T_i$. On the left-hand side of the energy equation put $\rho = \rho_\infty + \rho_1$, $u \approx u_0$, $T_1 \sim x^{-1/4}$, $K = K_\infty$. On the right-hand side put the values u_1, v_1, $T = T_0$, $K = K_\infty$, $\rho = \rho_\infty + \rho_1$. The boundary conditions for T_1 are similar to those for u_1. One obtains

$$J(c_v \rho_\infty u_0 T_{1x} - K_\infty T_{1yy}) = -R\rho_\infty T_0 u_{1x}. \tag{18}$$

Introduce a non-dimensional coordinate $\bar{\eta}$:

$$\begin{aligned} \bar{\eta} &= y\left(\frac{c_v u_0 \rho_\infty}{4K_\infty x}\right)^{1/2} = B\eta, \qquad B = \left(\frac{c_v \mu_\infty}{K_\infty}\right)^{1/2}; \\ \bar{\eta}_x &= -\frac{1}{2} x^{-1} \bar{\eta}, \qquad \bar{\eta}_y = \left(\frac{c_v u_0 \rho_\infty}{4K_\infty x}\right)^{1/2}; \\ \bar{T}_1 &= A_3 T_0 x^{-1/4} \bar{h}_1(\bar{\eta}), \qquad A_3 = \left(\frac{R}{Jc_v}\right) A_1 (\sim \text{in.}^{1/4}). \end{aligned} \tag{19}$$

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*In order to verify whether the value $v_1 = 0$ fulfills the second equation of motion to the first degree of approximation, put into this equation on the left-hand side the values: $\rho = \rho_\infty$, $u = u_0$, $v = v_1$, $\mu = \mu_\infty$, and on the right-hand side the values: $p_y = 0$, $v = 0$, $u = u_1$. Retain only the terms of degree $x^{-5/4}$. One obtains: $\rho_\infty u_0 v_{1x} - (4/3)\mu_\infty v_{1yy} = 0$. Hence $v_1 = 0$ is really a solution.

**It was assumed that $\rho_1 \sim \rho_0$.

After substituting (19) into (18) and performing the necessary transformations one obtains:

$$\bar{h}_1'' + 2\bar{\eta}\bar{h}_1' + \bar{h}_1 = 4[(4\eta^2 - 1)k_1 - 2\eta k_1'] \exp(-\eta^2). \tag{20}$$

Putting

$$\bar{h}_1 = \bar{p}_1(\bar{\eta}) \exp(-\bar{\eta}^2),$$

one gets

$$(\bar{p}_1'' - 2\bar{\eta}\,\bar{p}_1' - \bar{p}_1) \exp(-\bar{\eta}^2) = 4[(4\eta^2 - 1)k_1 - 2\eta k_1'] \exp(-\eta^2). \tag{21}$$

One complementary function of this equation is $\bar{k}_1$;* hence put $\bar{p}_1 = \bar{k}_1\bar{r}_1$ and obtain

$$[2(\bar{k}_1' - \bar{\eta}\bar{k}_1)\bar{r}_1' + \bar{k}_1\bar{r}_1''] \exp(-\bar{\eta}^2) = \bar{F}_{11}, \tag{22}$$

or

$$\frac{d}{d\bar{\eta}}[\bar{k}_1^2\bar{r}_1' \exp(-\bar{\eta}^2)] = \bar{F}_{11}\bar{k}_1 . \tag{23}$$

Hence

$$\bar{k}_1^2\bar{r}_1' \exp(-\bar{\eta}^2) = \int_0^{\bar{\eta}} \bar{F}_{11}\bar{k}_1\, d\bar{\eta} + C = \bar{F}_{12} . \tag{24}$$

As one may easily notice, after the integration is performed the expression on the right-hand side will consist of the following functions: the error (probability) integral and $\sum_0^\infty \bar{\eta}^{(2n+1)} \exp(-c\bar{\eta}^2)$. The constant of integration must be such as to fulfill the boundary condition $\bar{r}_1' = 0$ for $\bar{\eta} = 0$. In the next step one obtains

$$\bar{r}_1 = \int_0^{\bar{\eta}} \bar{k}_1^{-2}\bar{F}_{12} \exp(\bar{\eta}^2)\, d\bar{\eta} + C_1 . \tag{25}$$

The constant of integration may be different from zero to fulfill the boundary condition for $\bar{\eta} = 0$.** The function $\bar{k}_1^{-2}$ may be expanded into a series by ordinary division. The function $\bar{r}_1$ will consist of the following functions: $\sum_0^\infty \bar{\eta}^{2n} \exp(-c\bar{\eta}^2)$ and $\int_0^{\bar{\eta}} \bar{\eta}^n \operatorname{erf}(a\bar{\eta}) \exp(c\bar{\eta}^2)d\bar{\eta}$. This last integral is not tabulated and must be calculated in each particular case by numerical methods. One term in $\bar{r}_1$ will consist of an ordinary exponential function, which is not equal to zero for $\bar{\eta} = 0$.*** Hence all the boundary conditions are fulfilled and the final result is

$$\overline{T_1} = A_3T_0x^{-1/4}\bar{k}_1\bar{r}_1 \exp(-\bar{\eta}^2), \tag{26}$$

$$T_1 = A_3T_0x^{-1/4}k_1r_1 \exp(-B^2\eta^2). \tag{27}$$

C_1 will be determined from the condition that with no heat dissipation from the jet the total heat content (enthalpy) expressed as mass $\times i$ = mass $\times c_p(T - T_\infty) = 2c_p \int_0^\infty u_1T_1(\rho_\infty + \rho_1)dy$ = const., in each cross section.

*All the functions not included in the text are given in the Appendix.

**In the present case it will be assumed that $C_1 \neq 0$ (see below).

***Hence $\bar{r}_1(0) \neq 0$. But the value $T_1(0)$ will include C_1 (see below).

(e). *Coefficients of viscosity and heat conductivity.* Assuming that μ can be expressed as $\mu = \bar{\mu}(1 + aT + bT^2 + \cdots)$ and K as $K = K_0 c_p \mu$, and substituting the value $T = T_\infty + T_1$, one obtains

$$\mu_1 = \mu_\infty + \bar{\mu} \sum_1^n G_{2i} T_1^i , \qquad K_1 = K_0 c_p \mu_1 . \tag{28}$$

5. Second approximation.

(a) *Longitudinal velocity component.* Put into the first equation of motion on the left-hand side the values $\rho = \rho_\infty + \rho_1$, $u \approx u_0 + u_2$, $u_x = u_{1x} + u_{2x}$, $u_{yy} = u_{1yy} + u_{2yy}$, $\mu = \mu_1$, and on the right-hand side the values $p_x = R(\rho T)_x$, $T = T_\infty + T_1$, $v = 0$ and $u_2 \sim x^{-1/2}$. In this approximation only the terms of degree $x^{-3/2}$ will be retained:

$$u_0 \rho_\infty u_{2x} - \mu_\infty u_{2yy} = \bar{\mu} G_{21} (T_1 u_{1y})_y - u_0 \rho_1 u_{1x} - R(\rho_1 T_1)_x . \tag{29}$$

All the functions have to be taken as functions of η. The solution of this equation will follow precisely the procedure outlined in the calculation of T_1. The main points are the following:

$$u_2 = u_0 x^{-1/2} f_2(\eta), \qquad f_2 = k_2 \exp\left[-\tfrac{1}{2}\eta^2\right], \tag{30}$$

$$u_0^2 \rho_\infty x^{-3/2} [f_2'' + 2(\eta f_2' + f_2)] = u_0^2 \rho_\infty x^{-3/2} [k_2'' + (1 - \eta^2) k_2] \exp\left[-\tfrac{1}{2}\eta^2\right] = F_2 . \tag{31}$$

One complementary function is $f_{21} = 1 + \sum_{i=2,4,\ldots}^\infty b_i \eta^i$. Hence put $k_2 = f_{21} g_2$:

$$u_0^2 \rho_\infty x^{-3/2} (2 f_{21}' g_2' + f_{21} g_2'') \exp\left[-\tfrac{1}{2}\eta^2\right] = F_2 , \qquad \frac{d}{d\eta} (f_{21}^2 g_2') = \sum_1^2 F_{2i} ,$$

$$g_2' = f_{21}^{-2} \left[\int_0^\eta \sum F_{2i} \, d\eta + A_8 \right]. \tag{32}$$

Because in the expression on the right-hand side of (31) there will appear terms of the form $\exp(-\eta^2)$, the constant A_8 must be chosen so as to make $g_2'(0) = 0$, where

$$g_2 = \int_0^\eta \left[f_{21}^{-2} \left(\int_0^\eta \sum F_{2i} \, d\eta + A_8 \right) \right] d\eta + A_9 . \tag{33}$$

Since the function g_2 will contain terms of the form $\exp(-\eta^2)$, which do not vanish for $\eta = 0$, the constant of integration A_9 must be calculated and is not equal to zero.* The result is

$$u_2 = u_0 x^{-1/2} f_{21} g_2 \exp\left[-\tfrac{1}{2}\eta^2\right], \tag{34}$$

with all the boundary conditions fulfilled.

(b) *Transverse velocity component.* To verify the statement made above that $v_2 = 0$, put into the second equation of motion on the left-hand side the values $\rho = \rho_\infty + \rho_1$, $u \approx u_0 + u_2$, $v = v_2$, $\mu = \mu_1$, and on the right-hand side the values $p_y = R(\rho T)_y$, $u = u_1 + u_2$, $v = 0$, $\mu = \mu_1$, and keep only the terms of degree $x^{-3/2}$:

$$3 u_0 \rho_\infty v_{2x} - 4 \mu_\infty v_{2yy} = 0. \tag{35}$$

Hence $v_2 = 0$ is a solution.

*The rate M with the previous value of A_1 and with $\rho = \rho_\infty + \rho_1$, $u = u_1 + u_2$ must be constant in each cross section. This will give the value of A_9.

(c) *Density.* Put into the equation of continuity on the left-hand side the values $u \approx u_0 + u_2$, $\rho = \rho_\infty + \rho_1 + \rho_2$, and on the right-hand side the values $\rho = \rho_0 + \rho_1$, $u = u_1 + u_2$, $v = 0$. Preserving only the terms of degree $x^{-3/2}$, one obtains

$$u_0 \rho_{2x} = -\rho_0 u_{2x} - \rho_1 u_{1x}, \tag{36}$$

$$\rho_2 = -\tfrac{1}{2}\rho_0 x^{-1/2}\{2f_{21}g_2 \exp[-\tfrac{1}{2}\eta^2] - A_1^2 k_1^2 \exp(-2\eta^2)\}, \tag{37}$$

with the constant of integration equal to zero and with all the boundary conditions fulfilled.

(d) *Temperature.* Put into the equation of energy on the left-hand side the values $\rho = \rho_\infty + \rho_1 + \rho_2$, $u \approx u_0 + u_2$, $T = T_\infty + T_1 + T_2$, $K = K_1$, and on the right-hand side the values $\rho = \rho_\infty + \rho_1 + \rho_2$, $T = T_\infty + T_1$, $u = u_1 + u_2$, $v = 0$, $\mu = \mu_1$. Retaining only the terms of degree $x^{-3/2}$ gives:

$$\begin{aligned} J(c_v u_0 \rho_\infty \overline{T}_{2x} - K_\infty \overline{T}_{2yy}) &= J\overline{K}G_{21}(\overline{T}_1\overline{T}_{1y})_y + \mu_\infty \overline{u}_{1y}^2 \\ &\quad - Jc_v u_0 \overline{\rho}_1 \overline{T}_{1x} - R[\rho_\infty(\overline{u}_{1x}\overline{T}_1 + \overline{u}_{2x}T_\infty) + \overline{u}_{1x}\overline{\rho}_1 T_\infty], \end{aligned} \tag{38}$$

where bars over the letters denote the functions of $\overline{\eta}$. Again the procedure is identical with that explained above:

$$\begin{gathered} \overline{T}_2 = T_0 x^{-1/2}\overline{h}_2(\overline{\eta}), \qquad \overline{h}_2 = \overline{p}_2 \exp[-\tfrac{1}{2}\overline{\eta}^2], \\ A_{10}x^{-3/2}[\overline{h}_2'' + 2(\overline{\eta}\overline{h}_2' + \overline{h}_2)] = A_{10}x^{-3/2}[\overline{p}_2'' + (1 - \overline{\eta}^2)\overline{p}_2] \exp[-\tfrac{1}{2}\overline{\eta}^2] = \overline{G}_2 . \end{gathered} \tag{39}$$

With a complementary function $\overline{f}_{21}$* the procedure identical to that in Sec. 5(a) gives

$$\overline{f}_{21}^2 \overline{r}_2' = \sum_{i=1}^{6} B_i \overline{I}_{2i} + B_7 . \tag{40}$$

The constant of integration B_7 must be so chosen as to make $\overline{r}_2'(0) = 0$, where

$$\overline{r}_2 = \int_0^{\overline{\eta}} \left(\sum_{i=1}^{6} B_i \overline{I}_{2i} + B_7 \right) \overline{f}_{21}^{-2} \, d\overline{\eta} + B_8 . \tag{41}$$

The constant B_8 is not equal to zero, hence $\overline{r}_2(0) \neq 0$.** Hence:

$$\begin{gathered} \overline{T}_2 = T_0 x^{-1/2}\overline{f}_{21}\overline{r}_2 \exp[-\tfrac{1}{2}\overline{\eta}^2], \\ T_2 = T_0 x^{-1/2} f_{21} r_2 \exp[-\tfrac{1}{2}B^2\eta^2], \end{gathered} \tag{42}$$

with all the boundary conditions fulfilled.

(e) *Coefficients of viscosity and heat conductivity.*

$$\begin{gathered} \mu_2 = \mu_\infty + \overline{\mu}[G_{21}(T_1 + T_2) + G_{31}(T_1^2 + T_2^2) + G_{32}T_1T_2 + \cdots], \\ K_2 = K_0 \, c_p \, \mu_2 . \end{gathered} \tag{43}$$

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*The function $\overline{f}_{21}$ is the function f_{21} with η changed to $\overline{\eta}$.

**B_8 will be determined from the condition that

$$\text{mass } i = 2c_p \int_0^\infty (\rho_\infty + \rho_1 + \rho_2)(u_1 + u_2)(T_1 + T_2)\, dy = \text{const.},$$

in each cross section.

6. Third approximation.

(a) *Longitudinal velocity component.* Putting on the left-hand side of Eq. (4) the values $\rho = \rho_\infty + \rho_1 + \rho_2$, $u \approx u_0 + u_2 + u_3$, $u_x = u_{1x} + u_{2x} + u_{3x}$, $\mu = \mu_2$, $u_{yy} = u_{1yy} + u_{2yy} + u_{3yy}$, and on the right-hand side the values $p_x = R(\rho T)_x$, $v = 0$, etc., and letting $u_3 \sim x^{-3/4}$ gives the result:

$$\rho_\infty u_0 u_{3x} - \mu_\infty u_{3yy} = F_3 . \tag{44}$$

The procedure remains unchanged:

$$u_3 = u_0 x^{-3/4} f_3 , \tag{45}$$

$$\begin{gathered} u_0^2 \rho_\infty x^{-7/4}(f_3'' + 2\eta f_3' + 3f_3) = K_3 , \\ f_3 = k_3 \exp[-\tfrac{1}{2}\eta^2], \qquad u_0^2\rho_\infty x^{-7/4}[k_3'' + (2 - \eta^2)k_3] \exp[-\tfrac{1}{2}\eta^2] = K_3 . \end{gathered} \tag{46}$$

One complementary function is f_{31}. Put $k_3 = f_{31} g_3$:

$$\begin{gathered} u_0^2 \rho_\infty x^{-7/4} \frac{d}{d\eta} (f_{31}^2 g_3') \exp[-\tfrac{1}{2}\eta^2] = f_{31} K_3 , \\ f_{31}^2 g_3' = \sum_1^6 I_{3i} + C_6 , \qquad g_3 = \int_0^\eta \Big(\sum_1^6 I_{3i} + C_6 \Big) f_{31}^{-2}\, d\eta + C_7 . \end{gathered} \tag{47}$$

C_6 must be chosen so as to give $g_3'(0) = 0$. C_7 must be calculated and is not equal to zero. The result is:

$$u_3 = u_0 x^{-3/4} f_{31} g_3 \exp[-\tfrac{1}{2}\eta^2]. \tag{48}$$

(b) *Transverse velocity component.* Put on the left-hand side of Eq. (5) the values $\rho = \rho_\infty + \rho_1 + \rho_2$, $u \approx u_0 + u_1 + u_2$, $v = v_3 \sim x^{-3/4}$, $\mu = \mu_2$, and on the right-hand side the values $p_y = R(\rho T)_y$, $u = u_1 + u_2 + u_3$, $v = 0$. Retaining only the terms of degree $x^{-7/4}$ gives the result:

$$3\rho_\infty u_0 v_{3x} - 4\mu_\infty v_{3yy} = \mu_\infty u_{1xy} . \tag{49}$$

One may easily show that $v_3 = v_{t1}$, given by (16), is a solution of (49). In this case $u_{1x} = -v_{3y}$* or $u_{1xy} = -v_{3yy}$, hence Eq. (49) changes to $\rho_\infty u_0 v_{3x} - \mu_\infty v_{3yy} = 0$. But $v_3 = -\int u_{1x}\, dy$,** or $v_{3x} = -\int u_{1xx}\, dy$ and $v_{3yy} = -\int u_{1xyy}\, dy$. Hence after substituting into the simplified equation (49), i.e. $\rho_\infty u_0 \int u_{1xx}\, dy - \mu_\infty \int u_{1xyy}\, dy$, differentiating with respect to y and integrating with respect to x, one obtains exactly Eq. (8). But $v_3 = v_{t1}$ does not fulfill the boundary conditions when $y \to \infty$, viz., $g_{t1} \neq 0$ for $y \sim \eta \to \infty$. One has to find a solution of the homogeneous equation (49). To this end put

$$v_3 = u_0 x^{-3/4} p_3 , \tag{50}$$

and obtain $p_3'' + (3/2)\eta p_3' + (9/4)p_3 = 0$. One distinct solution of (50) is the series p_{31}; another[2] is

$$p_{32} = p_{31} \int_0^\eta p_{31}^{-2} \exp[-\tfrac{3}{4}\eta^2]\, d\eta + C_8 . \tag{51}$$

*See Sec. 4(b).

**The constant of integration equals 0.

[2]See: E. L. Ince, *Ordinary differential equations*, Longmans, Green and Co., New York, 1927, p. 122.

An integration term by term may be applied. C_8 must be chosen so as to make $(g_{t1} + p_{32}) = 0$ as $\eta \to \infty$.

7. **Final remarks.** The approximations of higher degree are only a matter of routine work. In the theory of incompressible fluids equations of motion and continuity are used. The velocity components may be derived from the stream function. But it is almost impossible always to satisfy all the boundary conditions for all the velocity components. Usually the v — component satisfies only one condition (for $\eta = 0$ but not for $\eta \to \infty$). It was shown above that in the theory of compressible flow one may always satisfy all the boundary conditions by use of the distinct solutions of homogeneous equations. This last remark refers equally to all the dependent variables.*

APPENDIX

4(a). $k_1 = 1 + a_2\eta^2 + a_4\eta^4 + \cdots + a_{2n}\eta^{2n} + \cdots, \quad a_2 = 1/2, \quad a_4 = 5/24,$

$$a_6 = 1/16, \quad a_8 = 13/896, \quad a_{10} = 221/80640, \quad \text{etc.};$$

$$k_{12} = 1 + \alpha_4\eta^4 + \alpha_8\eta^8 + \cdots, \quad \alpha_4 = 1/12, \quad \alpha_8 = 1/672, \quad \text{etc.}$$

4(d). $\bar{k}_1 = 1 + \sum_{i=2,4,\cdots}^{\infty} a_i\bar{\eta}^i, \quad \bar{F}_{11} = 4[(4B^{-2}\bar{\eta}^2 - 1)\bar{k}_1 - 2\bar{\eta}\bar{k}_1'] \exp(-B^{-2}\bar{\eta}^2).$

4(e). $\mu_\infty = \bar{\mu}(1 + aT_\infty + bT_\infty^2 + cT_\infty^3 + \cdots), \quad G_{21} = a + 2bT_\infty + 3cT_\infty^2 + \cdots,$

$$G_{22} = b + 3cT_\infty + \cdots, \quad \text{etc.}; \quad K_\infty = K_0c_p\mu_\infty .$$

5(a). $F_2 = 4[R(\rho_1T_1)_x + u_0\rho_1u_{1x} - \bar{\mu}G_{21}(u_{1y}T_1)_y], \quad b_2 = -1/2, \quad b_4 = 1/8,$

$$b_6 = -1/48, \quad b_8 = 1/384, \quad b_{10} = -7/34560, \quad \text{etc.};$$

$$F_{21} = \sum_1^4 F_{21i}f_{21} \exp[-(B^2 + 1/2)\eta^2], \quad F_{211} = \{2(A_5 + A_7) - 4[A_5(1 + B)$$

$$+ A_7]\eta\}k_1^2r_1, \quad F_{212} = 4(A_5 + A_7)k_1k_1'r_1\eta, \quad F_{213} = (2A_5k_1r_1'\eta - A_7k_1''r_1)k_1,$$

$$F_{214} = -A_7[(k_1r_1)_\eta - 2B^2\eta k_1r_1](k_1' - 2\eta k_1), \quad F_{22} = A_6[2\eta k_1' + (1 - 4\eta^2)k_1]k_1f_{21}$$

$$\cdot \exp[(-3/2)\eta^2], \quad A_5 = A_1A_3Ru_0^{-2}T_0(\rho_0/\rho_\infty)(\text{inch}^{1/2}),$$

$$A_6 = A_1^2(\rho_0/\rho_\infty)(\text{inch}^{1/2}), \quad A_7 = A_1A_3G_{21}T_0\bar{\mu}\mu_\infty^{-1}(\text{inch}^{1/2}).$$

5(d). $\bar{K} = K_0c_p\bar{\mu}, \quad A_{10} = Jc_vu_0\rho_\infty T_0, \quad \bar{G}_2 = 4\{R[\rho_\infty(\bar{u}_{1x}\bar{T}_1 + \bar{u}_{2x}T_\infty) + \bar{u}_{1x}\bar{\rho}_1T_\infty]$

$$+ J[c_vu_0\bar{\rho}_1\bar{T}_{1x} - \bar{K}G_{21}(\bar{T}_1\bar{T}_{1y})_y] - \mu_\infty\bar{u}_{1y}^2\}; \quad B_1 = A_1A_3R(Jc_v)^{-1}(\text{inch}^{1/2}),$$

$$B_2 = RT_\infty(Jc_vT_0)^{-1}(\text{dimensionless}), \quad B_3 = A_1^2R\rho_0T_\infty(Jc_v\rho_\infty T_0)^{-1}(\text{inch}^{1/2}),$$

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*This fact has been shown by the author in several papers.

$$B_4 = A_1 A_3 \rho_0 \rho_\infty^{-1}(\text{inch}^{1/2}), \qquad B_5 = A_3^2 G_{21} \overline{K} T_0 \rho_0 (K_\infty \rho_\infty)^{-1}(\text{inch}^{1/2});$$

$$\overline{I}_{21} = \int_0^{\bar\eta} [(4B^{-2}\bar\eta^2 - 1)\bar k_1 - 2\bar k_1'\bar\eta]\bar k_1 \bar r_1 \bar f_{21} \exp[-(B^{-2} + 1/2)\bar\eta^2]\, d\bar\eta,$$

$$\overline{I}_{22} = 2\int_0^{\bar\eta} [(B^{-2}\bar\eta^2 - 1)\bar f_{21}\bar g_2 - \bar\eta(\bar f_{21}\bar g_2)_{\bar\eta}]\bar f_{21} \exp[(-1/2)(B^{-2} - 1)\bar\eta^2]\, d\bar\eta,$$

$$\overline{I}_{23} = \int_0^{\bar\eta} [2\bar k_1'\bar\eta + (1 - 4B^{-2}\bar\eta^2)\bar k_1]\bar f_{21}\bar k_1 \exp[-(2B^{-2} - 1/2)\bar\eta^2]\, d\bar\eta,$$

$$\overline{I}_{24} = \int_0^{\bar\eta} [\bar\eta(\bar k_1 \bar r_1)_{\bar\eta} - (4\bar\eta^2 - 1)\bar k_1 \bar r_1]\bar f_{21}\bar k_1 \exp[-(B^{-2} + 1/2)\bar\eta^2]\, d\bar\eta,$$

$$\overline{I}_{25} = -\int_0^{\bar\eta} \{(\bar k_1\bar r_1)_{\bar\eta\bar\eta} - 4\bar\eta(\bar k_1\bar r_1)_{\bar\eta} + (4\bar\eta^2 - 1)\bar k_1\bar r_1 + [(\bar k_1\bar r_1)_{\bar\eta} - 2\bar\eta\bar k_1\bar r_1]^2\}\bar f_{21}$$
$$\cdot \exp[-3\bar\eta^2/2]\, d\bar\eta,$$

$$\overline{I}_{26} = -\int_0^{\bar\eta} (\bar k_1' - 2B^{-2}\bar\eta\bar k_1)^2 \bar f_{21} \exp[-(2B^{-2} - 1/2)\bar\eta^2]\, d\bar\eta;$$

$$G_{31} = b + 3cT_\infty + \cdots, \qquad G_{32} = 2b + 6cT_\infty + \cdots$$

6(a). $$F_3 = \bar\mu[G_{21}(u_{1y}T_2 + u_{2y}T_1) + G_{31}u_{1y}T_1^2]_y - u_0(u_{1x}\rho_2 + u_{2x}\rho_1) - \rho_\infty u_{1x}u_2$$
$$- R(\rho_1 T_2 + \rho_2 T_1)_x, \qquad K_3 = -4F_3; \qquad f_{31} = 1 + \sum_{i=2,4,\cdots}^{\infty} d_i \eta^i,$$

$$d_2 = -1, \quad d_4 = 1/4, \quad d_6 = -1/20, \quad d_8 = 1/160, \quad d_{10} = -1/1440,$$

etc.;

$$I_{31} = -2C_1 \int_0^{\eta} [(\eta^2 - 1)f_{21}g_2 - \eta(f_{21}g_2)_\eta]f_{31}k_1 \exp(-\eta^2)\, d\eta,$$

$$I_{32} = (-1/2)C_1 \int_0^{\eta} [(4\eta^2 - 1)k_1 - 2\eta k_1']\{2f_{21}g_2 - A_1^2 k_1^2 \exp[-3\eta^2/2]\}f_{31}$$
$$\cdot \exp(-\eta^2)\, d\eta,$$

$$I_{33} = A_1 \int_0^{\eta} [(4\eta^2 - 1)k_1 - 2\eta k_1']f_{21}f_{31}g_2 \exp(-\eta^2)\, d\eta,$$

$$I_{34} = 2C_2 \int_0^{\eta} \{A_1 k_1 f_{21} r_2 \exp[-(1 + B^2/2)\eta^2] + (1/2)\, A_3(2f_{21}g_2 - A_1^2 k_1^2$$

$$\cdot \exp{[-3\eta^2/2]})k_1 r_1 \exp{[-(B^2 + 1/2)\eta^2]}\}_\eta f_{31} \exp{[\eta^2/2]}\eta \, d\eta,$$

$$I_{35} = -\int_0^\eta \{C_3(k_1' - 2\eta k_1)f_{21}r_2 \exp{[-(1 + B^2/2)\eta^2]}$$

$$+ C_4[(f_{21}g_2)_\eta - \eta f_{21}g_2]k_1 r_1 \exp{[-(B^2 + 1/2)\eta^2]}\}_\eta f_{31} \exp{[\eta^2/2]} \, d\eta,$$

$$I_{36} = -C_5 \int_0^\eta \{(k_1' - 2\eta k_1)k_1^2 r_1^2 \exp{[-(2B^2 + 1)\eta^2]}\}_\eta f_{31} \exp{[\eta^2/2]} \, d\eta;$$

$$C_1 = A_1(\rho_0/\rho_\infty)(\text{inch}^{1/4}), \qquad C_2 = R\rho_0 T_0(u_0^2\rho_\infty)^{-1}(\text{dimensionless}),$$

$$C_3 = A_1 G_{21} T_0 \bar{\mu}\mu_\infty^{-1}(\text{inch}^{1/4}), \qquad C_4 = A_3 G_{21} T_0 \bar{\mu}\mu_\infty^{-1}(\text{inch}^{1/4}),$$

$$C_5 = A_1 A_3^2 G_{31} T_0^2 \bar{\mu}\mu_\infty^{-1}(\text{inch}^{3/4}).$$

6(b). $p_{31} = C_8\left(1 + \sum_{i=2,4,\cdots}^{\infty} e_i \eta^i\right), \qquad e_2 = -9/8, \qquad e_4 = 63/128,$

$e_6 = -693/5120, \qquad e_8 = 6237/229376, \qquad e_{10} = -39501/9175040, \qquad$ etc.

—NOTES—

ON TYPES OF CONVERGENCE AND ON THE BEHAVIOR OF APPROXIMATIONS IN THE NEIGHBORHOOD OF A MULTIPLE ROOT OF AN EQUATION*

By E. BODEWIG (*The Hague*)

1. Introduction. A chief problem of practical analysis consists in computing zeros of functions. For this, successive approximations are used. There are procedures in which the initial estimate is arbitrary and will always yield the nearby zero-point as the limit of a sequence of approximations. But there are only a few such procedures, the method of Laguerre being one. In almost all other cases the first estimate must satisfy some conditions in order that the sequence of approximate values will converge. How to obtain a useful first approximation is a special problem and has nothing to do with the method of approximation itself.

Now there are *two gaps in the theory of all methods of approximations. First,* it is only proved that the method will converge; there is no possibility of comparing different methods of solving the same problem. Until now, one was satisfied to demonstrate the procedures by one or several examples and, from these special examples, general conclusions about the method were drawn. Such conclusions are false, of course.

Second, it is supposed that the convergence of the same method will be similar in all circumstances. The problem was not even stated as to whether the convergence could depend on the multiplicity of the zero which is to be approximated. On the contrary, one assumed tacitly that this would not be the case, which is false again.

Both gaps occur for the same reason, namely that until now only the notion of convergence was introduced, with no measure of the convergence. With such a measure both problems can be solved in principle: several approximation methods can be compared, and it can be decided if the character of the convergence of the same method will always be the same.

2. A measure of the convergence of a sequence. A measure for the rapidity of the convergence of a sequence has first been given by the author.[1]

Definition. A convergent sequence

$$x_1\,, x_2\,, x_3\,, \cdots, x_n\,, \cdots \to X$$

with the limit X is called "convergent in the degree g" if

$$\lim_{n\to\infty} \frac{x_{n+1} - X}{(x_n - X)^g} = c \neq 0.$$

Introducing the deviations of the terms from their limit, $x_n - X = d_n$, this becomes

$$\lim \frac{d_{n+1}}{d_n^g} = c \neq 0.$$

To determine the degree of convergence we have, therefore, to develop d_{n+1} in

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[1] E. Bodewig, *Sur la méthode de Laguerre*, Akad. Wetensch. Proc. (Amst.) **49**, 911-21, especially 912 (1946).

powers of d_n. The lowest term cd_n^g of this development determines the degree g and the coefficient c.

The larger g is, the more rapidly the sequence will converge. Briefly we can say that, if k is the number of exact decimals of x_n, then the number of exact decimals of x_{n+1} will be kg. Whatever the coefficient c may be, therefore, a method of approximation will be more effective, in the long run, than any other method with a lower degree of convergence. Only in the beginning of the sequence may the influence of the coefficient c change this behavior.

For the reasons given in the introduction, the author believes the notion of degree of convergence to be of fundamental importance in numerical analysis.

The reciprocal of a linear operator. We give first a general example and seek a process which will yield the reciprocal of a linear operator by a sequence of successive approximations converging in degree k.

For this purpose we consider first such a process for the reciprocal of a number N. Here $X = 1/N$, and by our definition we must have

$$d_{n+1} = cd_n^k + \cdots,$$

or

$$x_{n+1} - 1/N = c(x_n - 1/N)^k + \cdots.$$

The constant c is still arbitrary. It has to be chosen in such a manner that all powers of the unknown $1/N$ will be cancelled. This is done by letting $c = N^{k-1}$. Thus we have

$$x_{n+1} = x_n\left[k - \binom{k}{2} z_n + \binom{k}{3} z_n^2 - \cdots \pm z_n^{k-1}\right], \qquad \text{where} \qquad z_n = Nx_n .$$

The sequence $x_1, x_2, \cdots$ will converge if

$$0 < x_1 < 2/N,$$

and the convergence will be of degree k.

For $k = 2$ the known formula

$$x_{n+1} = x_n(2 - Nx_n)$$

results, which converges quadratically.

The same formula will also hold for any linear operator, since the proof did not use any properties other than those which are true for every linear operator. But the conditions of convergence must be established for every operator separately.

So the formulas hold if N is a matrix. This case has been investigated by G. Schulz[2] and again by other authors independently. In a former paper[3] the author has given another proof of the formulas slightly different from the above proof.

3. Newton's formula and generalizations. First we give a generalization of Newton's formula. While the usual Newton formula is customarily obtained by means of Taylor's theorem, we shall proceed here in another way.

[2]G. Schulz, *Iterative Berechnung einer reziproken Matrix*, Zeits. angew Math. Mech. **13**, 57-59 (1933).

[3]E. Bodewig, *Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten*, Akad. Wetensch. Proc. (Amst.) **50**, 930-941, 1104-1116, 1285-1295 (1947) and **51**, 53-64, 211-219 (1948).

Let $f(x)$ have the root X of multiplicity p:

$$f(x) = (x - X)^p \cdot g(x), \qquad \text{where} \qquad g(X) \neq 0. \tag{1}$$

Then

$$f'(x) = p(x - X)^{p-1}g + (x - X)^p g'$$

or

$$\frac{f'(x)}{f(x)} = \frac{p}{x - X} + \frac{g'(x)}{g(x)} . \tag{2}$$

Near $x = X$ the first term on the right is very large, while the second one is "finite". So with very good approximation,

$$\frac{f'(x)}{f(x)} \approx \frac{p}{x - X} ,$$

that is

$$X \approx x - p\,\frac{f(x)}{f'(x)} .$$

Thus *the general Newton's formula in the neighborhood of a root of multiplicity* **p** *is*

$$x_{n+1} = x_n - p\,\frac{f(x_n)}{f'(x_n)} . \tag{3}$$

More generally, we set

$$x_{n+1} = x_n - a\,\frac{f(x_n)}{f'(x_n)} \tag{3'}$$

and determine the character of the convergence of the sequence

$$x_1 , x_2 , x_3 , \cdots , x_n , \cdots \to X,$$

which we suppose converges, the conditions on x_1 being unknown. Developing $g(x)$ at $x = X$ we obtain from (1) that

$$\begin{aligned} f(x_n) &= (x_n - X)^p g(X) + (x_n - X)^{p+1} g'(X) + \cdots \\ &= d_n^p g(X) + d_n^{p+1} g'(X) + \cdots \end{aligned}$$

or by (3′),

$$x_{n+1} = x_n - \frac{a}{p}\,d_n + \frac{a}{p^2}\,\frac{g'(X)}{g(X)}\,d_n^2 + \cdots .$$

Subtracting X from both sides, we get

$$d_{n+1} = \left(1 - \frac{a}{p}\right)d_n + \frac{a}{p^2}\,\frac{g'(X)}{g(X)}\,d_n^2 + \cdots .$$

Here the linear term of d_n will be cancelled only if $a = p$. So we have the following results.

Result 1. In the neighborhood of a root of multiplicity p formula (3′) will yield a sequence which converges quadratically to the root (if it converges at all) only for $a = p$. The convergence is better the larger p is.

2. For $a \neq p$, however, the sequence is only linearly convergent; the smaller the quotient a/p is, the worse the convergence. So Newton's formula will not give better convergence in this case than, for instance, the rule of false position, which is often simpler.

3. The usual Newton formula (having $a = 1$) will therefore converge quadratically only near a simple root, but linearly near a multiple root.

4. If it is known that the root is multiple (for instance by observing that $f'(x_1)$ is small) but the exact multiplicity is not known, we put $a = 2$ or perhaps $a = 3$.

A formula which always yields quadratic convergence, without requiring knowledge of the multiplicity of the roots, is obtained by choosing a function which has all the roots of $f(x)$, with each of them simple. This is f/f'. Then (3′) with $a = 1$ will yield

$$x_{n+1} = x_n - \frac{ff'}{f'^2 - ff''}, \tag{4}$$

where on the right side the argument x_n has been omitted. The calculation is simpler if we put (4) in the form

$$x_{n+1} = x_n - \frac{f}{f' - A} \qquad \text{with} \qquad A = ff''/f'. \tag{4'}$$

More generally we put

$$x_{n+1} = x_n - \frac{ff'}{f'^2 - aff''} \tag{5}$$

and determine the character of the convergence for the case that

$$f(x) = (x - X)^p g(x).$$

Proceeding as above, we obtain

$$d_{n+1} = c_1 d_n + c_2 d_n^2 + \cdots,$$

where

$$c_1 = -(p-1)(a-1)/(p + a - ap), \qquad c_2 = -(a + ap - p)g'(X)/p(p + a - ap)^2 g(X).$$

So c_1 vanishes in the cases $a = 1$ and $p = 1$. The first case yields formula (4). In the second case also c_2 will vanish if $a = 1/2$. So we have the results below.

Result 1. Formula (4) yields a convergence which is always quadratic. Since the factor c_2 is equal to $g'(X)/pg(X)$, the convergence is least good in the case of a simple root; the greater the multiplicity p of the root, however, the better the convergence.

2. The quadratic convergence of (4) for a root of multiplicity p is worse than that of (3) if $p \neq 1$ and is the same if $p = 1$.

3. The simple formula (3) is therefore always preferable to (4), though only applicable if p is known.

4. In the case of a simple root, formula (5), with a arbitrary, will always yield quadratic convergence. It is, therefore, no better than the usual Newton formula. The Newton formula, moreover, is preferable as it does not contain f''.

5. Formula (5) with $a = 1/2$ yields *cubic convergence* in the case of a simple root, with

$$c_3 = -\frac{1}{2}\frac{g''(X)}{g(X)} + \left[\frac{g'(X)}{g(X)}\right]^2,$$

but only linear convergence with $c_1 = (p-1)/(p+1)$ in the case of a multiple root. Hence it is no better a formula than the simple Newton formula would be.

6. A convergence which is always cubical, is obtained by taking the average of the results of formulas (3) and (4). For the coefficients c_2 are absolutely equal, but of opposite sign in both formulae.

The simplified Newton method. It is known that the usual Newton formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ is often simplified to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_1)} \tag{6}$$

with constant denominator. Some authors (for instance Whittaker-Robinson) consider formula (6) to be nearly as effective as the usual one. This is, however, not the case. For it is easily seen that every sequence

$$x_{n+1} = x_n - cf(x_n)$$

yields $d_{n+1} = d_n - cf(x_n) = d_n - cd_n^p g(X) + \cdots$, and so always converges linearly for every p.

The same result has been given by Ostrowski.[4] The present proof is much simpler. It must be added, however, that for Ostrowski the chief point is to give the conditions for convergence.

Example. To illustrate the difference between quadratic and cubic convergence we take an example of H. S. Wall.[5] The square root of b is calculated by means of Newton's formula and by means of (5). Taking $f(x) = x^2 - b$ we obtain the two formulas:

$$x_{n+1} = \tfrac{1}{2}(x_n + b/x_n) \qquad \text{with} \qquad c_2 = 1/2b^{1/2},$$

$$x_{n+1} = (x_n^2 + 3b)x_n/(3x_n^2 + b) \qquad \text{with} \qquad c_3 = 1/4b.$$

Putting $b = 2$ and $x_1 = 1$ we obtain the sequences where all figures are correct and only the last ones are slightly different:

	quadratic convergence	cubic convergence
x_1	1	1
x_2	1.5	1.40
x_3	1.417	1.4142132
x_4	1.414216	1.414213562373095048796
x_5	1.414213562375	

[4] A. Ostrowski, *Über eine Modifikation des Newtonschen Näherungsverfahrens*, Akad. Nauk. SSSR (Gruz, Fil Trudy) 2, 241-249 (1937).

[5] H. S. Wall, *A modification of Newton's method*, Amer. Math. Monthly 55, 90-91 (1948).

Starting from $x_1 = 10$ we have the sequences:

	quadratic convergence	cubic convergence
x_1	10	10
x_2	5	3.5
x_3	2.7	1.6
x_4	1.7	1.415
x_5	1.44	1.414213564

4. Laguerre's method for algebraic equations and related formulas. While Newton's method and the generalized methods above are applicable to all equations and hold for real and complex roots, Laguerre[6] has given an elegant method which applies only to algebraic equations having all real roots. The sequence is defined by

$$x_{k+1} = x_k - \frac{nf(x_k)}{f'(x_k) \pm [H(x_k)]^{1/2}}$$

where n is the degree of the equation $f(x) = 0$ and $H(x) \equiv (n-1)^2 f'^2 - n(n-1)ff'' =$ Hesse's covariant of f. The sign of the square root can be taken as either $+$ or $-$, for the method approximates two roots simultaneously.

The method has been analyzed in an earlier paper of the author.[7] The features of the method are:

1. Laguerre proved that the method always converges, i.e. x_1 is arbitrary (in contrast to Newton's method, for example, wherein x_1 must satisfy certain conditions).

2. Laguerre proved that each sign of the square root yields a convergent sequence:

$$x_1\ , x_2'\ , x_3'\ , \cdots , x_k'\ , \cdots \to X'$$

$$x_1\ , x_2''\ , x_3''\ , \cdots , x_k''\ , \cdots \to X''$$

where X' is the nearest root lying to the left of x_1 and X'' the nearest root lying to the right of x_1. This will always hold if the straight line is considered closed at infinity.

3. The author has proved that the convergence is cubic in the case of a simple root, but only linear in the case of a multiple root.

4. In order to obtain cubic convergence in the case of a root of multiplicity p, the author proved that in the above formula, $H(x_k)$ is to be replaced by $aH(x_k)$ where $a = (n-p)/p(n-1)$.

5. Later van der Corput[8] proved that if the equation has at least one root of multiplicity $\geqq p$, the formula mentioned which replaces $H(x)$ by $aH(x)$ will approximate only the two roots of multiplicity $\geqq p$ lying to the left and to the right of x_1, thus skipping the roots of multiplicity $<p$. In the case of a root of multiplicity p the convergence is cubic, whereas in the case of a root of multiplicity $>p$ it is only linear.

5. The general convergence of degree n. In the preceding some important examples

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[6]E. N. Laguerre, *Sur une méthode pour obtenir par approximation les racines d'une équation algébrique qui a toutes ses racines réelles*, Oeuvres de Laguerre **1**, 87-103 (1898).

[7]See footnote 1.

[8]J. G. van der Corput, *Sur l'approximation de Laguerre des racines d'une équation algébrique qui a toutes ses racines réelles*, Akad. Wetensch. Proc. (Amst.) **49**, 922-929 (1946).

have been given showing the difference in convergence near a simple and a multiple root. This will now be generalized.

For this purpose we suppose we have a process $F(x)$ yielding to every value x_k a better value x_{k+1}:

$$x_{k+1} = F(x_k). \tag{7}$$

We are not concerned here with the conditions for convergence itself of the sequence, but merely with the character of the convergence. Now we seek the general function $F(x)$ which yields a sequence converging in degree n. Then

$$x_{k+1} - X = c(x_k - X)^n + \cdots . \tag{8}$$

Now when $D^i \equiv d^i/dx^i$ is the ith derivate,

$$D^iF(x_k) = D^ix_{k+1} = cn(n-1) \cdots (n-i+1)(x_k - X)^{n-i} + \cdots .$$

Putting $x_k = X$, the right side will vanish if $i < n$. So

$$D^iF(X) = 0 \qquad \text{for} \qquad i = 1, 2, \cdots, n-1. \tag{9}$$

This is the modified condition that the sequence $x_1, x_2, \cdots$ converges in degree n.

Before determining $F(x)$ we conclude from (8) that

$$X = \lim x_{k+1} = \lim F(x_k) = F(\lim x_k) = F(X),$$

that is

$$X = F(X). \tag{10}$$

This condition holds for every root X of $f(x) = 0$. Therefore

$$F(x) = x + f(x)g(x), \tag{11}$$

where $g(x)$ is arbitrary.

Now if (9) is to hold for $i = 1$, the sequence x_k must converge quadratically. Therefore, since

$$0 = DF(X) = 1 + f'(X)g(X),$$

$$g(X) = -1/f'(X) \qquad \text{or} \qquad g(x) = -1/f'(x) + f(x)g_1(x)$$

and the function (11) becomes

$$F(x) = x - f(x)/f'(x) + f^2(x)g_1(x). \tag{12}$$

It will yield a quadratically convergent sequence for an arbitrary $g_1(x)$.

Now if (9) has to hold for $i = 2$, that is if the sequence has to converge cubically, we obtain

$$0 = D^2F(X) = -D^2(f/f') + D^2(f^2g_1).$$

Here f^2g_1 is differentiated by Leibnitz' rule, and in the result x is replaced by X. Then all terms will vanish, except the last one: $g_1D^2(f^2) = g_1(X)\cdot 2f'^2(X)$. On the other hand since $f(X) = 0$, we have for $x = X$, if for brevity we set $r = 1/f'$,

$$D^2(f/f') = D^2(fr) = 2Df\cdot Dr + rD^2f = -2f'f''r^2 + f''r = -f''r.$$

Therefore

$$0 = D^2F(X) = f''(X)\cdot r(X) + 2f'^2(X)g_1(X),$$

or

$$g_1(x) = -1/2\, f''(x)r^3(x) + f(x)g_2(x)$$

and

$$F(x) = x - fr - 1/2\, f^2f''r^3 + f^3g_2\,, \tag{13}$$

where $g_2(x)$ is arbitrary.

To obtain generally *a sequence converging in degree n* we generalize (12) and (13) by means of the operator

$$P = r\cdot D, \qquad \text{where} \qquad r = 1/f'(x) \qquad \text{and} \qquad D = d/dx$$

and write

$$F(x) = F_n(x) - f^n(x)g_n(x), \tag{14}$$

where

$$F_n(x) = x - fr + \frac{1}{2} f^2\cdot Pr - \frac{1}{6} f^3\cdot P^2r + \cdots + \frac{(-1)^{n-1}}{(n-1)!} f^{n-1}\cdot P^{n-2}r. \tag{14a}$$

We have to prove that this is the expression we obtain if we proceed from (13) to sequences which converge in degree 4, 5, $\cdots$, n. Since the form of F_2 and F_3 is identical with (12) and (13), respectively, we proceed by induction and assume that $F_n(x)$ is the true expression; we have to prove that then $F_{n+1}(x)$ is the expression yielding a sequence which converges in degree $n + 1$.

Now since

$$D(fr) = r\,Df + f\,Dr = 1 + ff'\cdot Pr,$$

$$\begin{aligned} DF_n(x) = 1 &- (1 + ff'\cdot Pr) + (ff'\cdot Pr + \tfrac{1}{2}f^2f'\cdot P^2r) - \tfrac{1}{6}(3f^2f'\cdot P^2r \\ &+ f^3f'\cdot P^3r) + \cdots + \frac{(-1)^{n-1}}{(n-1)!}[(n-1)!f^{n-2}f'\cdot P^{n-2}r + f^{n-1}f'\cdot P^{n-1}r] \\ = &\ \frac{(-1)^{n-1}}{(n-1)!} f^{n-1}f'\cdot P^{n-1}r. \end{aligned}$$

Then since

$$D^m[f(X)]^m = m![f'(X)]^m,$$

it follows that

$$D^nF_n(X) = D^{n-1}DF_n(X) = (-1)^{n-1}[f'(X)]^{n-1}f'(X)\cdot P^{n-1}r.$$

Furthermore,

$$D^n\{[f(X)]^ng_n(X)\} = n![f'(X)]^ng_n(X),$$

that is

$$g_n(X) = \frac{1}{n!} P^{n-1} r \qquad \text{q.e.d.}$$

Although (14) and (14a) have been constructed to yield a sequence converging in degree n (if it converges at all), we have to investigate what happens *when the root X is multiple.* In this case the equation $0 = DF(X)$ standing between Eqs. (11) and (12) can no longer be satisfied by any function $g(x)$ whatever. For we must have

$$0 = DF(X) = 1 + f'(X)g(X) = 1,$$

since $f'(X) = 0$ and X is multiple. So not even a quadratic convergence can be obtained, that is, *the convergence is linear.*

The formula (14) can be given another form which originally dates from Euler.[9] He gave the following generalization of the method of Newton about which the author has reported in a former paper.[10]

For the function $y = f(x)$ a value X has to be determined such that $y = 0$. Using the inverse function $x = u(y)$, X is given by $X = u(0)$. But u is unknown, and of X only an approximate value a is known. Thus, $f(a)$ is known. Now, for $A = f(a)$ Euler develops $u(0)$ in a Taylor series near A:

$$X = u(A - A) = u(A) - Au'(A) + 1/2A^2u''(A) - 1/6\, A^3u'''(A) + \cdots . \tag{15}$$

Here the derivates of u for the argument A are known, for

$$u(A) = a, \qquad u'(A) = dx/dy = 1/f'(a) = r(a),$$

$$u''(A) = -f''(a)r^3(a), \qquad u'''(A) = [3f''^2(a) - f'(a)f'''(a)]r^5(a), \cdots .$$

Setting

$$u^{(n)}(y) = (-1)^{n-1}U_n(x)r^{2n-1}(x),$$

we have the recurrence formula for U_n :

$$U_{n+1} = (2n - 1)f''U_n - f'U_n' . \tag{16}$$

Now the first terms of (15) are identical with the corresponding terms of (14a). This must be proved true for all the terms. Thus we proceed again by induction and suppose that (15) holds for n and prove that it will hold for $n + 1$. Let therefore

$$P^n r = u^{(n)}(y), \qquad \text{then}$$

$$P^{n+1}r = PP^n r = Pu^{(n)} = r \cdot Du^{(n)} = (-1)^{n-1}r[U_n'r^{2n-1} - (2n - 1)f''U_n r^{2n}]$$

$$= (-1)^n U_{n+1}r^{2n+1} = u^{(n+1)}(y), \qquad \text{q.e.d.}$$

So we have the results following.

Result 1. The developments (14) and (15) are identical.

2. The convergence of the sequence x_1 , x_2 , $\cdots$ yielded by the function $F(x)$ of (14) and (14a) is of degree n if the root X to which it converges is simple, but only linear if the root is multiple.

[9] L. Euler, *Institutiones Calculi Differentialis*, II, Cap. IX.—Opera Omnia, ser. I, vol. X, p. 422-455.

[10] E. Bodewig, *Über das Eulersche Verfahren zur Auflösung numerischer Gleichungen*, Commentarii Math. Helvetici **8**, 1-4 (1935).

ON THE COMPRESSION OF A SHORT CYLINDER BETWEEN ROUGH END-BLOCKS*

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1. In this paper we consider non-uniform elastic compression between parallel end-blocks of a cylindrical test specimen with circular cross-section. It is assumed that there is sufficient friction between the end-blocks and the end faces of the specimen to prevent slippage. Thus, points on the end faces of the cylinder undergo *only* axial displacements.

It is the purpose of this paper to determine the relation between the force F acting on the specimen, the compression $2a$, and Young's modulus E of the material, assuming Poisson's ratio ν to be known. If lateral expansion were not obstructed by the friction at the end surfaces, these quantities would be connected by the relation $E = Fh/Aa$, where $2h$ denotes the height of the specimen, b its radius and A its cross-sectional area. If there is friction at the end surfaces, a correction must be applied to this simple formula: the derivation of this correction will now be sketched.

The above problem is solved to a good approximation by finding upper and lower bounds for the strain energy of the specimen, using the method of Prager and Synge.[1]

It is advantageous to resolve the original problem, which we shall denote by P_0, into two simpler problems P_c and P, where P_c is the problem of simple compression and P is that problem which will yield the initial problem P_0 when superimposed on P_c.

In the natural cylindrical coordinates, the original problem P_0 involves the boundary conditions

$$\begin{aligned} u_r(r, \pm h) &= 0, \qquad & u_z(r, \pm h) &= \mp a, \\ T_n(b, z) &= 0, \qquad & T_t(b, z) &= 0, \end{aligned} \tag{1}$$

where u_r and u_z are the displacements in the r and z directions respectively and T_n, T_t are the normal and tangential components of stress transmitted through the surface.

Thus the problem P, with which we shall deal from now on, has the boundary conditions

$$\begin{aligned} u_r(r, \pm h) &= -\nu ar, \qquad & u_z(r, \pm h) &= 0, \\ T_n(b, z) &= 0, \qquad & T_t(b, z) &= 0. \end{aligned} \tag{2}$$

The analysis used in the following is quite similar to that given in a previous paper[2] in which the case of plane strain was considered, except that here we have cylindrical coordinates and different boundary conditions. The similarity, however, permits the procedure and results alone to be indicated here.

2. Since the bar possesses rotational symmetry, the displacements u_r and u_z are functions of r and z only and not of θ, and there is no circumferential displacement, i.e. $u_\theta = \partial u/\partial\theta = 0$.

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[1]W. Prager and J. L. Synge, *Approximations in elasticity based on the concept of function space*, Q. Appl. Math. **5**, 241–269 (1947).

[2]H. J. Greenberg and R. Truell, *On a problem in plane strain*, Q. Appl. Math. **6**, 53 (1948).

Hooke's law then has the form

$$\left.\begin{aligned} e_{rr} &= \frac{1}{E}\,[E_{rr} - \nu(E_{\theta\theta} + E_{zz})], \\ e_{zz} &= \frac{1}{E}\,[E_{zz} - \nu(E_{rr} + E_{\theta\theta})], \\ e_{\theta\theta} &= \frac{1}{E}\,[E_{\theta\theta} - \nu(E_{rr} + E_{zz})], \\ e_{rz} &= \frac{1+\nu}{E}\,E_{rz}\,, \\ e_{r\theta} &= e_{z\theta} = 0, \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} E_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\,[(1-\nu)e_{rr} + \nu(e_{\theta\theta} + e_{zz})], \\ E_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\,[(1-\nu)e_{zz} + \nu(e_{rr} + e_{\theta\theta})], \\ E_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\,[(1-\nu)e_{\theta\theta} + \nu(e_{rr} + e_{zz})], \\ E_{rz} &= \frac{E}{1+\nu}\,e_{rz}\,, \qquad E_{r\theta} = E_{z\theta} = 0. \end{aligned}\right\} \tag{4}$$

The strains are defined in terms of the displacements by

$$\left.\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r}, \qquad & e_{zz} &= \frac{\partial u_z}{\partial z}, \\ e_{\theta\theta} &= \frac{u_r}{r}, \qquad & e_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \end{aligned}\right\} \tag{5}$$

The equilibrium equations reduce to

$$\frac{\partial}{\partial r}\,(rE_{rr}) + \frac{\partial}{\partial z}\,(rE_{rz}) = E_{\theta\theta}\,, \qquad \frac{\partial}{\partial r}\,(rE_{rz}) + \frac{\partial}{\partial z}\,(rE_{zz}) = 0. \tag{6}$$

The Eqs. (6) are satisfied automatically if we choose our stresses to be

$$\begin{aligned} E_{rr} &= \frac{\psi}{r}, \qquad & E_{zz} &= -\frac{1}{r}\frac{\partial \varphi}{\partial r}, \\ E_{\theta\theta} &= \frac{\partial \psi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2}, \qquad & E_{rz} &= \frac{1}{r}\frac{\partial \varphi}{\partial z}, \end{aligned} \tag{7}$$

where ψ and φ are two arbitrary functions, called the *stress functions*.

Following the Prager-Synge notation and development we have the following boundary conditions:

a) On $\mathbf{S}^*$, the completely associated state,

$$u_r^*(r, \pm h) = -\nu ar, \qquad u_z^*(r, \pm h) = 0; \tag{8}$$

b) On $\mathbf{S}'_p$, the homogeneous associated states,

$${}_pu'_r(r, \pm h) = 0, \qquad {}_pu'_z(r, \pm h) = 0; \tag{9}$$

c) On $\mathbf{S}''_q$, the complimentary states,

$${}_qE''_{rr}(b, z) = 0, \qquad {}_qE''_{rz}(b, z) = 0, \tag{10}$$

where the stresses ${}_qE''_{ij}$ must satisfy the equilibrium conditions (6).

3. If $\mathbf{S}_0$, $\mathbf{S}_c$ and $\mathbf{S}$ are the "natural" states of the problems P_0, P_c and P respectively, then it can be seen at once that the strain energies can be superimposed by simple addition.

Thus

$$S_0^2 = S_c^2 + S^2 \tag{11}$$

since the boundary conditions of P_c and P are such that the scalar product $\mathbf{S}_c \cdot \mathbf{S} = \int T_i^c u_i \, ds$ vanishes. Finding bounds on S_0^2 therefore reduces to finding bounds on S^2 since S_c^2 is known to be $2\pi a^2 E$.

Prager-Synge give the following inequality for S^2:

$$\sum_{q=1}^{n} (\mathbf{S}^* \cdot \mathbf{I}''_q)^2 \le S^2 \le S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^* \cdot \mathbf{I}'_p)^2 \tag{12}$$

where $\mathbf{I}'_p$, $\mathbf{I}''_q$ are obtained by orthonormalizing the sequences $\mathbf{S}'_p$, $\mathbf{S}''_q$ respectively. As in footnote 2, this reduces to

$$\frac{2}{\pi(U_m + 2)} \frac{F}{a} \le E \le \frac{2}{\pi(L_n + 2)} \frac{F}{a}, \tag{13}$$

where

$$L_n = \frac{1}{\pi a^2 E} \sum_{q=1}^{n} (\mathbf{S}^* \cdot \mathbf{I}''_q)^2, \qquad U_m = \frac{1}{\pi a^2 E} \left[S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^* \cdot \mathbf{I}'_p)^2 \right]. \tag{14}$$

4. Suitable displacement functions, satisfying appropriate boundary conditions and obvious symmetry conditions, are chosen to obtain the states $\mathbf{S}^*$ and $\mathbf{S}'_p$. Stress functions φ and ψ are selected for the computation of the states $\mathbf{S}''_q$. The orthonormalization is carried out using the Gram-Schmidt procedure.

The bounds U_m and L_n are computed at each iteration and after five steps we obtain the inequality

$$0.29958 \frac{F}{a} \le E \le 0.30371 \frac{F}{a},$$

where we have now set $b = h = 1$ and $\nu = 1/3$.

Averaging, we have

$$E = 0.3017 \frac{F}{a}. \tag{15}$$

In terms of the mean stress $\sigma = F/\pi b^2$ and the mean strain $\epsilon = a/h$, we obtain

$$E = 0.9477 \frac{\sigma}{\epsilon}. \tag{16}$$

Let us now perform the test in which the specimen is in a state of compression, such as in problem P_0, and measure the applied force F and the compression $2a$. The formula

$$\bar{E} = \frac{Fh}{Aa} = \frac{\sigma}{\epsilon} \tag{17}$$

no longer yields the true value of E since this formula applies exactly only to the test in which there is no friction at the end-blocks, i.e. simple compression. However the true value of E is given by Eq. (16) and the relation between E and $\bar{E}$ is found by comparison of (16) and (17), namely

$$E = 0.9477\,\bar{E}. \tag{18}$$

This gives us the correction factor to be applied to Eq. (17) to yield the true value of Young's modulus E. The numerical coefficient of $\bar{E}$ in (18) is in error by less than 0.7%.

The displacements and strains of the completely associated state and the homogeneous associated states are given in Table I below. The stress functions and stresses of the complementary states are shown in Table II.

TABLE I. The Associated States

	u'_r	u'_z	e'_{rr}	$e'_{\theta\theta}$	e'_{zz}	e'_{rz}
S'_1	0	$-(1-z^2)z$	0	0	$-(1-3z^2)$	0
S'_2	$rz^2(1-z^2)$	0	$z^2(1-z^2)$	$z^2(1-z^2)$	0	$rz(1-2z^2)$
S'_3	0	$-r^2(1-z^2)z$	0	0	$-r^2(1-3z^2)$	$-rz(1-z^2)$
S'_4	$r^2z^2(1-z^2)$	0	$2rz^2(1-z^2)$	$rz^2(1-z^2)$	0	$r^2z(1-2z^2)$
S'_5	0	$-r(1-z^2)z$	0	0	$-r(1-3z^2)$	$-\frac{1}{2}z(1-z^2)$
S^*	$-\frac{1}{3}rz^4$	0	$-\frac{1}{3}z^4$	$-\frac{1}{3}z^4$	0	$-\frac{2}{3}rz^3$

TABLE II. The Complementary States.

	φ	ψ	E''_{rr}	$E''_{\theta\theta}$	E''_{zz}	E''_{rz}
S''_1	$\frac{1}{2}z^4r^2(1-r^2)$	0	0	$6z^2r^2(1-r^2)$	$z^4(2r^2-1)$	$2z^3r(1-r^2)$
S''_2	0	$z^2r(1-r^2)$	$z^2(1-r^2)$	$z^2(1-3r^2)$	0	0
S''_3	$\frac{1}{2}r^2$	0	0	0	-1	0
S''_4	0	$z^2r^2(1-r^2)$	$z^2r(1-r^2)$	$2z^2r(1-2r^2)$	0	0
S''_5	$\frac{1}{3}r^3$	0	0	0	$-r$	0

RIGOROUS SOLUTION OF A DIFFERENTIAL EQUATION IN SOIL MECHANICS*

By R. GRAN OLSSON (*Norwegian Institute of Technology, Trondheim*)

In one of his earlier books K. Terzaghi[1] published a differential equation describing approximately the progress of consolidation in a sediment which is being deposited at a constant rate q per unit of time. In connection with his solution of this equation Terzaghi pointed out that it does not satisfy the boundary conditions for time $t = 0$. In the following paragraphs the writer presents a solution which satisfies the boundary conditions for any time, $t = 0$ included.

Let $a_v[gm^{-1}cm^2]$ be the coefficient of compressibility, C a constant of integration, c [sec] a time constant, $k[cm\ \sec^{-1}]$ the coefficient of permeability (Darcy's coefficient), $q[gm\ cm^{-2}\ \sec^{-1}]$ the quantity of sedimentation per unit of area and time, t [sec] the time, γ_1[gm cm^{-3}] the unit weight, $\gamma_w[gm\ cm^{-3}]$ the unit weight of water $= 1$ $[gm\ cm^{-3}]$, γ $[gm\ cm^{-3}]$ the submerged unit weight $(=\gamma_1 - \gamma_w)$, ζ the time function, $z = (c/t)^{1/2}$ a dimensionless independent variable, and $\phi(z)$ the error integral.

Using these symbols, which are identical with those used by Terzaghi, except that γ_w represents the unit weight of water, the differential equation[2] assumes the form

$$\frac{d\zeta}{dt} + \frac{3}{t^2}\left(\frac{t}{2} + \frac{c}{3}\right)\zeta = \frac{c}{t^2}, \tag{1}$$

where $c = 3\gamma^2 k/\gamma_w\, aq^2$ is a constant with the dimension time.

With

$$z = (c/t)^{1/2}$$

then

$$t = cz^{-2}$$

and

$$dt = -2cz^{-3}\,dz, \qquad \frac{d\zeta}{dt} = -\frac{z^3}{2c}\frac{d\zeta}{dz};$$

substituting these values into Eq. (1), we obtain the equation

$$\frac{d\zeta}{dz} = \left(\frac{3}{z} + 2z\right)\zeta - 2z. \tag{1a}$$

This equation is solved in the usual way by first finding the complementary function from the homogenous equation

$$\frac{d\zeta}{\zeta} = \left(\frac{3}{z} + 2z\right)dz.$$

Integrating we obtain

$$\zeta(z) = Cz^3 \exp(z^2)$$

where C is an arbitrary constant.

*Received Oct. 27, 1948.

[1] K. Terzaghi, *Erdbaumechanik auf bodenphysikalischer Grundlage*, Franz Deuticke, Leipzig and Vienna, 1924, p. 175.

[2] K. Terzaghi, op. cit., p. 175, Eq. (119).

The general solution is given as

$$\zeta(z) = C(z)z^3 \exp(z^2),$$

where $C(z)$ is an arbitrary function to be determined by (1a). Differentiating we get

$$\frac{d\zeta}{dz} = [C'(z)z^3 + (3z^2 + 2z^4)C(z)] \exp(z^2),$$

which inserted into (1a) yields,

$$C'(z) = -2z^{-2} \exp(-z^2).$$

By integration we finally get

$$C(z) = -2 \int \exp(-z^2)z^{-2}\, dz.$$

To find the integral we integrate by parts:

$$\int \exp(-z^2)z^{-2}\, dz = -\frac{\exp(-z^2)}{z} - 2\int \exp(-z^2)\, dz,$$

and thus

$$C(z) = \frac{2}{z}\exp(-z^2) + 4\int \exp(-z^2)\, dz + C_1,$$

where C_1 is again an arbitrary constant, which is determined by the boundary condition for $t = 0$.

It is convenient to introduce the error integral as a known (and tabulated) function[3],

$$\int \exp(-z^2)\, dz = \frac{1}{2}\pi^{1/2}\phi(z).$$

Thus we get for $C(z)$ the expression

$$C(z) = \frac{2}{z}\exp(-z^2) + 2\pi^{1/2}\phi(z) + C_1$$

and for the complete solution of the Eq. (1),

$$\zeta(z) = 2z^3 \exp(z^2)\left[z^{-1}\exp(-z^2) + \pi^{1/2}\phi(z) + \frac{1}{2}C_1\right].$$

The constant C_1 is determined from the boundary condition that for $t = 0$, $\zeta = \zeta_0 = 1$. When $t \to 0$, $z \to \infty$, we resort to the asymptotic development of $\phi(z)$ in a semiconvergent series

$$\phi(z) = 1 - \frac{\exp(-z^2)}{z}\left(1 - \frac{1}{2z^2} + \frac{1\cdot 3}{(2z^2)^2} - \frac{1\cdot 3\cdot 5}{(2z^2)^3} + - \cdots\right)$$

and get for $\zeta(z)$,

$$\zeta(z) = 2z^3 \exp(z^2)\left[\frac{1}{z}\exp(-z^2) + \pi^{1/2} - \frac{1}{z}\exp(-z^2)\left(1 - \frac{1}{2z^2}\right.\right.$$

$$\left.\left. + \frac{1\cdot 3}{(2z^2)^2} - + \cdots\right) + \frac{1}{2}C_1\right].$$

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[3]E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, New York, 1945, p. 24.

By putting $C_1 = -2\pi^{1/2}$ we get $\zeta = 1$ when $z \to \infty$, since both constant terms and z^{-1} exp (z^2) and $-z^{-1}$ exp (z^2) cancel. We finally obtain

$$\zeta(z)_{z\to\infty} = 2z^3 \exp(z^2)\left(\frac{\exp(-z^2)}{2z^3} - \frac{1\cdot 3}{4z^5}\exp(-z^2) + - \cdots\right) = 1 - \frac{3}{2z^2} + - \cdots .$$

This satisfies exactly the boundary condition $\zeta = 1$ for $t = 0$. The diagram shown in Fig. 1 gives ζ as a function of the non-dimensional "time" t/c. It is to be observed that

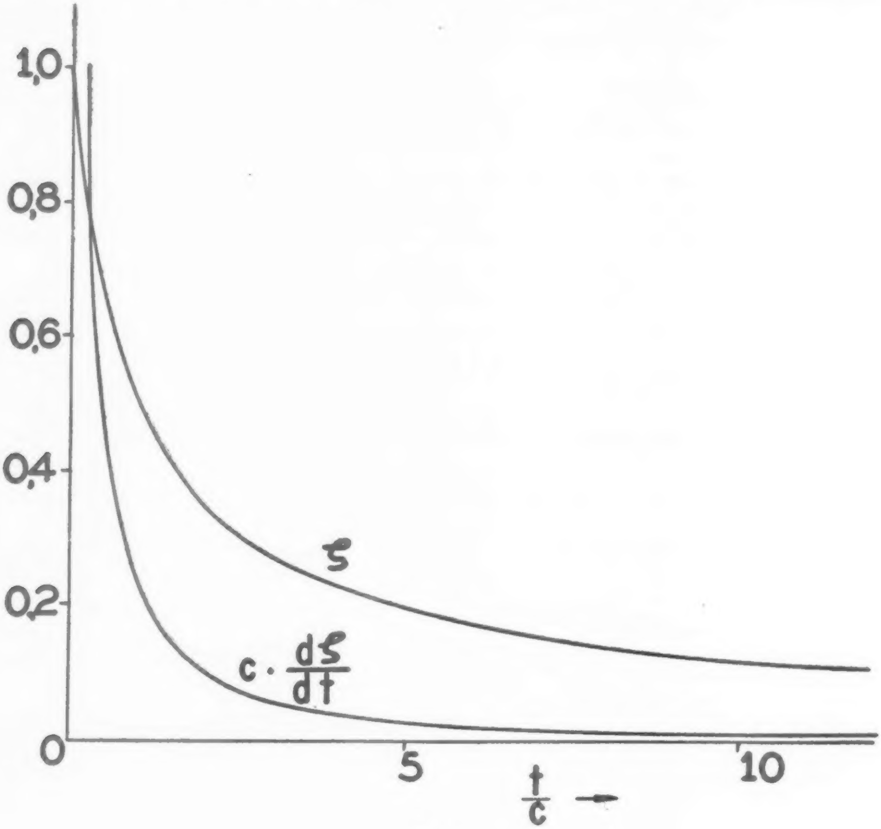

Fig. 1

for small values of t/c we must use very accurately tabulated values of $\phi(z)$ because the function ζ appears as the difference between two nearly equal quantities.[4] In general, the solution of the differential equation satisfying exactly the boundary condition $\zeta = 1$ for $t = 0$ will have the form

$$\zeta(z) = 2z^3 \exp(z^2)[z^{-1}\exp(-z^2) + \pi^{1/2}\phi(z) - \pi^{1/2}].$$

[4] J. Burgess, *On the definite integral* $(2/\pi^{1/2}) \int_0^t \exp(-t^2)\, dt$ *with extended tables.*, Trans. Roy. Soc. Edingurgh 39 (Part II), (1898)

This solution may be given a different form by introducing the derivative of the error function

$$\phi'(z) = \frac{2}{\pi^{1/2}} \exp(-z^2) \qquad \text{or} \qquad \exp(z^2) = \frac{2}{\pi^{1/2}} [\phi'(z)]^{-1}.$$

Thus we get

$$\zeta(z) = 4[\phi'(z)]^{-1} z^3 [\phi(z) + 2z^{-1}\phi'(z) - 1]$$

as the most elegant form in which the solution may be written.

Besides $\zeta(z)$ the derivative $d\zeta/dt$ is also of interest. From the Eq. (1a) we get

$$\frac{d\zeta}{dz} = 4z^3[\phi'(z)]^{-1}[\phi(z) + 2z^{-1}\phi'(z) - 1](3z^{-1} + 2z) - 2z.$$

Further we have

$$\frac{d\zeta}{dt} = \frac{d\zeta}{dz}\frac{dz}{dt} = -\frac{z^3}{2c}\frac{d\zeta}{dz}$$

and finally

$$\frac{d\zeta}{dt} = -\frac{2z^6}{c}[\phi'(z)]^{-1}[\phi(z) + 2z^{-1}\phi'(z) - 1](3z^{-1} + 2z) + \frac{z^4}{c}.$$

The diagram in Fig. 1 gives $d\zeta/dt$ as a function of the independent variable t/c.

APPENDIX. When the sedimentation process is finished, the time factor increases according to the equation [1, p. 176, Eq. (122)]

$$(1 - \zeta) \exp(2kt/a\gamma_w h_1^2) = (1 - \zeta_1) \exp(2kt_1/a\gamma_w h_1^2) \tag{1}$$

or by introducing the "time constant"

$$c = \frac{3\gamma^2 k}{\gamma_w a q^2} \tag{2}$$

into the Eq. (1) we get

$$(1 - \zeta) \exp(2ct/3t_1^2) = (1 - \zeta_1) \exp(2c/3t_1). \tag{1b}$$

The time is so determined, that t is equal to zero at the beginning of the sedimentation process and $t = t_1$ at its end. We obtain the value of ζ_1 from the earlier solution of the differential equation, i.e.,

$$\zeta_1 = 4\phi'(z_1)^{-1} z_1^3 \left[\phi(z_1) + \frac{1}{2z_1^2}\phi'(z_1) - 1\right], \tag{3}$$

by introducing $z_1 = (c^7/t_1)^{1/2}$ into the solution (3).

Equation (1b) determines a system of curves starting from a point $(c/t_1, \zeta_1)$ on the curve ζ and having the horizontal line $\zeta = 1$ as an asymptote. Thus we get the system of curves represented in Fig. 2. It can easily be proved that there exists only *one* set of such curves according to the Eqs. (1) and (1b), so that the curves in Fig. 2

are a general representation of the consolidation problem after finishing of the sedimentation process. In Fig. 2 the upper curves correspond to the greater values of k/a

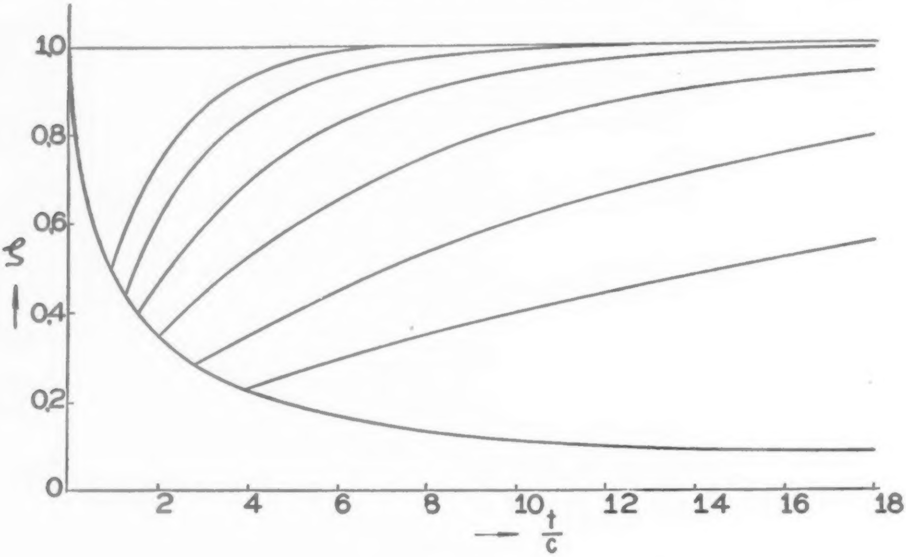

Fig. 2

as valid for sand and other permeable materials, whereas the lower curves correspond to clay and similar earth masses with smaller values of the coefficient of permeability.

NOTE ON THE PROBLEM OF TWISTING OF A CIRCULAR RING SECTOR*

By ERIC REISSNER (*Massachusetts Institute of Technology*)

The problem of twisting of a circular ring sector is of some interest in connection with the calculation of stresses and deformations in close-coiled helical springs. To be considered is a ring-sector under the action of two equal and opposite forces P along the axis through the center of the ring and perpendicular to the plane of the ring (Fig. 1). A formulation of the problem and an outline of results by O. Göhner for sectors of solid circular and rectangular cross section may be found on pp. 355-361 in *Theory of elasticity* by S. Timoshenko.

The purpose of the present note is to obtain explicit results for the twisting of ring sectors of hollow cross sections, with thin walls. Formulas will be obtained which have the same meaning for the present problem as R. Bredt's formulas have for the problem of St. Venant torsion of cylindrical rods.

The problem may be considered as one of the membrane theory of thin shells of

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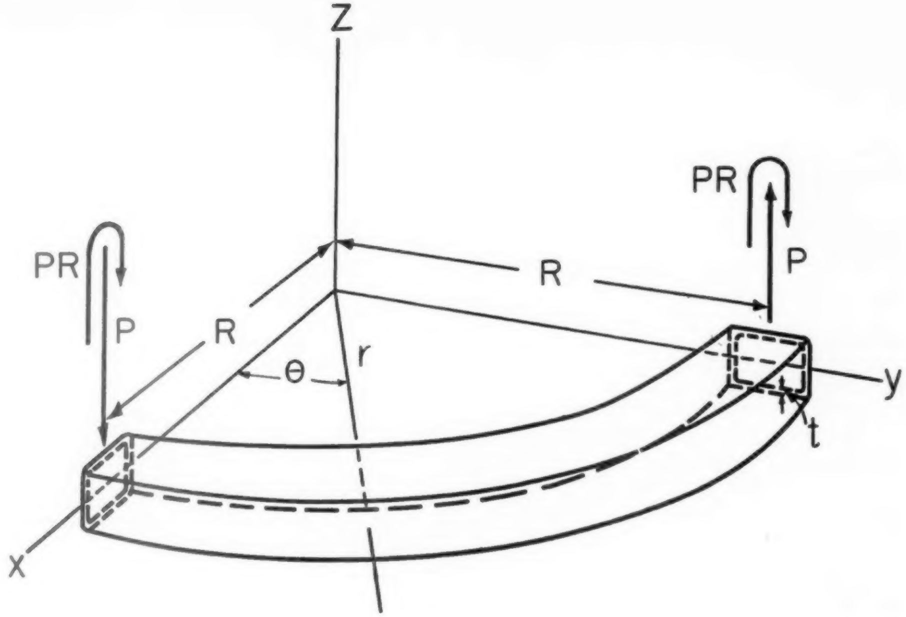

FIG. 1

revolution, with multi-valued expressions for displacements. A direct solution, without recourse to general shell theory may be obtained as follows. The assumptions of the twisting theory of ring sectors are equivalent to requiring that all stress resultants of the membrane theory vanish with the exception of the shear stress resultant S acting

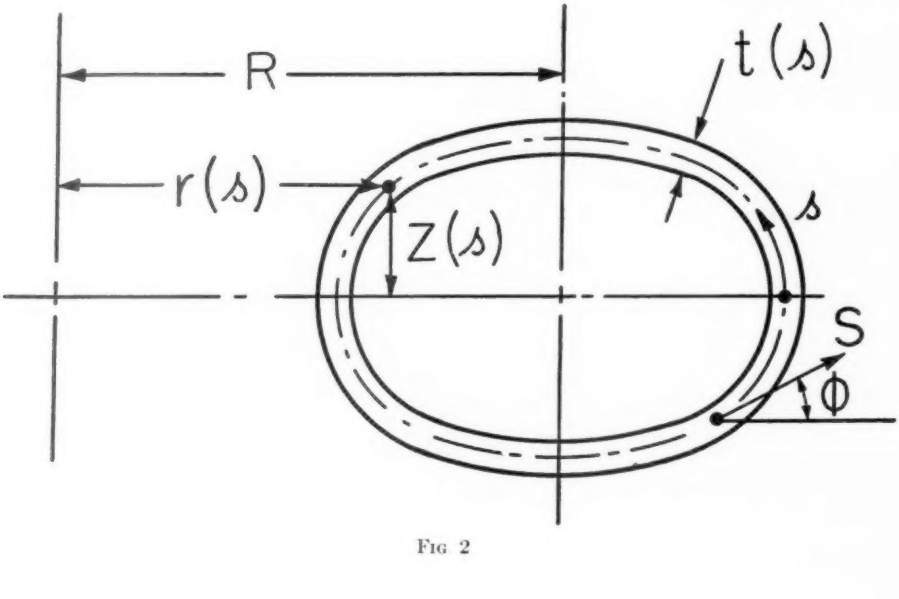

FIG 2

over the cross section of the ring sector. (Fig. 2). As all stresses possess rotational symmetry the resultant S satisfies the following equilibrium equation

$$\frac{d(rS)}{ds} + S\frac{dr}{ds} = 0, \tag{1}$$

where s is the arc length measured along the center line of the tube wall and r is the distance of the points of this line from the axis of the ring sector. From (1) it follows that

$$r^2 S = C. \tag{2}$$

Note that the largest value of the shear stress resultant S occurs at the point nearest the axis. From the relation

$$P = \oint S \sin\phi \, ds \tag{3}$$

the value of the constant C is obtained:

$$C = P/\oint \frac{dz}{r^2}. \tag{4}$$

Equation (2) may thus be written in the following form

$$S = \frac{P}{r^2 \oint (1/r^2)\, dz}. \tag{5}$$

Equation (5) is the counterpart of Bredt's formula $S = T/2A$ of torsion theory for the shear stress resultant S in terms of the twisting couple T.

For the determination of the deformation of the ring sector we use the stress-strain relation

$$S = Gt\gamma \tag{6}$$

where G is the modulus of rigidity, t the wall thickness and γ the appropriate shearing strain component. In terms of the components of strain in cylindridal coordinates $\gamma_{\theta r}$ and $\gamma_{\theta z}$ we have

$$\gamma = \gamma_{\theta r} \cos\phi + \gamma_{\theta z} \sin\phi. \tag{7}$$

In view of the fact that S is the only non-vanishing stress resultant, all other components of strain for cylindrical coordinates vanish,

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = 0. \tag{8}$$

Components of displacement in the radial, circumferential and axial directions which are compatible with (8) are*

$$U = 0, \qquad V = V(r, z), \qquad W = k\theta. \tag{9}$$

From (9) the remaining components of strain are obtained in the form

$$\begin{aligned} \gamma_{\theta r} &= r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) + \frac{1}{r}\frac{\partial U}{\partial \theta} = r\frac{\partial}{\partial r}\left(\frac{V}{r}\right), \\ \gamma_{\theta z} &= \frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial W}{\partial \theta} = \frac{\partial V}{\partial z} + \frac{k}{r}. \end{aligned} \tag{10}$$

*These expressions hold without any assumption concerning the form of the cross section of the ring sector.

Combination of (10) and (7) gives

$$\gamma = r \frac{\partial}{\partial r}\left(\frac{V}{r}\right)\frac{dr}{ds} + \frac{\partial V}{\partial z}\frac{dz}{ds} + \frac{k}{r}\frac{dz}{ds} \tag{11}$$

or

$$\frac{\gamma}{r} = \frac{d}{ds}\left(\frac{V}{r}\right) + \frac{k}{r^2}\frac{dz}{ds}. \tag{12}$$

In Eq. (12) one takes γ in terms of S from Eq. (6) and then integrates (12) over the closed cross section. In view of the fact that V must be a univalued function this leads to the following relation:

$$\oint \frac{S\,ds}{Gt\,r} = k \oint \frac{dz}{r^2}, \tag{13}$$

or, with S from (5),

$$\frac{k}{P} = \oint \frac{ds}{Gt\,r^3}\left\{\oint \frac{dz}{r^2}\right\}^{-2} \tag{14}$$

where according to (9) the change in length per winding of a spring has a value $2\pi k$. Equation (14) is the counterpart of the well-known Bredt formula $\Theta/T = [\oint(ds/Gt)]/4A^2$ for the twist-torque ratio for closed thin-walled sections.

Examples. We take for a first example the case of a tube with circular cross section and with uniform wall thickness. The equations of the center line of the tube wall are taken in the form

$$r = R + a\cos\psi, \qquad z = a\sin\psi. \tag{15}$$

The integrals occurring in Eqs. (5) and (14) become

$$\begin{aligned}\oint \frac{dz}{r^2} &= \frac{a}{R^2}\int_0^{2\pi} \frac{\cos\psi\,d\psi}{[1 + (a/R)\cos\psi]^2} \\ &= -\frac{2\pi}{R}\frac{(a/R)^2}{[1 - (a/R)^2]^{3/2}}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\oint \frac{ds}{r^3} &= \frac{a}{R^3}\int_0^{2\pi} \frac{d\psi}{[1 + (a/R)\cos\psi]^3} \\ &= \frac{\pi a}{R^3}\frac{2 + (a/R)^2}{[1 - (a/R)^2]^{5/2}}.\end{aligned} \tag{17}$$

The ratio (5) between stress resultant S and applied force P is then

$$\frac{S}{P} = \frac{R}{2\pi a^2}\frac{[1 - (a/R)^2]^{3/2}}{[1 + a/R\cos\psi]^2}. \tag{18}$$

As the ratio a/R tends to zero the tube stress resultant S approaches the value $S = PR/2\pi a^2 = PR/2A$ which coincides, as it should, with the value predicted by pure torsion theory for an applied couple of magnitude PR.

The maximum stress occurs when $\cos \psi = -1$ and is of the following magnitude

$$\frac{S_{max}}{P} = \frac{R}{2\pi a^2} \frac{[1 - (a/R)^2]^{3/2}}{[1 - (a/R)]^2} = \frac{R}{2\pi a^2}\left[1 + 2\frac{a}{R} + \frac{3}{2}\left(\frac{a}{R}\right)^2 + \cdots\right]. \tag{19}$$

The deflection-force ratio k/P as given by Eq. (14) becomes

$$\frac{k}{P} = \frac{1}{Gt}\frac{R^3}{2\pi a^3}\left[1 - \left(\frac{a}{R}\right)^2\right]^{1/2}\left[1 + \frac{1}{2}\left(\frac{a}{R}\right)^2\right] = \frac{1}{Gt}\frac{R^3}{2\pi a^3}\left[1 - \frac{3}{8}\left(\frac{a}{R}\right)^4 - \frac{1}{8}\left(\frac{a}{R}\right)^6 - \cdots\right]. \tag{20}$$

The absence of a term of the form $(a/R)^2$ inside the bracket indicates that the influence of the factor in brackets is quite small in all but extreme cases.

As a second example we consider a tube with rectangular cross section and uniform thickness. We designate by R the distance of the center of the cross section from the axis, by $2a$ the width of the tube and by $2b$ the height of the tube. We find that

$$\oint \frac{dz}{r^2} = -\frac{8ab}{R^3}\frac{1}{[1 - (a/R)^2]^2} \tag{21}$$

and

$$\oint \frac{ds}{r^3} = \frac{4(a + b)}{R^3}\frac{1 + (3b - a)(a/R)^2/(b + a)}{[1 - (a/R)^2]^3} \tag{22}$$

From (21) and (5) the maximum value of the stress resultant S follows in the form

$$\frac{S_{max}}{P} = \frac{R}{8ab}\frac{[1 - (a/R)^2]^2}{[1 - (a/R)]^2} = \frac{R}{8ab}\left[1 + 2\frac{a}{R} + \left(\frac{a}{R}\right)^2 + \cdots\right]. \tag{23}$$

It is noteworthy that the factor in brackets in (23) is independent of the height $2b$ of the tube cross section, and that this factor is somewhat smaller than the corresponding factor of Eq. (19) for the circular tube.

Combination of (14), (21) and (22) gives for the deflection-force ratio of the rectangular tube the following result

$$\frac{k}{P} = \frac{1}{Gt}\frac{(a + b)R^3}{16a^2b^2}\left[1 - \left(\frac{a}{R}\right)^2\right]\left[1 + \frac{3b - a}{b + a}\left(\frac{a}{R}\right)^2\right] = \frac{1}{Gt}\frac{(a + b)R^3}{16a^2b^2}\left[1 + 2\frac{b - a}{b + a}\left(\frac{a}{R}\right)^2 - \frac{3b - a}{b + a}\left(\frac{a}{R}\right)^4\right]. \tag{24}$$

A comparison with the corresponding formula (20) shows that the square tube shares with the circular tube the property that no terms with $(a/R)^2$ occur within the last bracket. In contrast to this, terms with $(a/R)^2$ do occur whenever one of the sides of

the rectangle is longer than the other. It may be of interest to list the following special cases. Denoting the factor in brackets by $1 + \delta$, we find that when

$$a \ll b, \qquad 1 + \delta \sim 1 + 2(a/R)^2 - 3(a/R)^4, \tag{25a}$$

and when

$$b \ll a, \qquad 1 + \delta \sim 1 - 2(a/R)^2 + (a/R)^4. \tag{25b}$$

It may be noted when $a \ll b$ then δ assumes a maximum value of 5/27 for $(a/R)^2 = 1/3$, whereas when $b \ll a$ then δ is always negative.

CORRECTIONS* TO THE PAPER

ON A CLASS OF SINGULAR INTEGRAL EQUATIONS OCCURRING IN PHYSICS

QUARTERLY OF APPLIED MATHEMATICS *6*, 443-448 (1949)

BY H. P. THIELMAN (*Iowa State College*)

The limits on the integral in Eq. (B), p. 445 have been omitted. They should have been indicated as 0 and ∞.

Equation (a) of Theorem I, p. 445 should read $kf(0) - f'(0) = 0$ and not $kf(0) - f''(0) = 0$ as stated. It should have been stated that $f''(x)$ in Theorem I, and $f^{iv}(x)$ in Theorem II are assumed to be of order $o(e^{kx})$ as x goes to infinity.

*Received June 6, 1949.

BOOK REVIEWS

Proceedings of a symposium on large-scale digital calculating machinery. Jointly sponsored by the Navy Department Bureau of Ordnance and Harvard University at the Computation Laboratory. Harvard University Press, Cambridge, Massachusetts, 1948. xxix + 302 pp. $10.00.

This is a collection of papers and discussions of papers presented at a symposium on large-scale digital computing machinery held at Harvard University on January 7-10, 1947. The meeting was sponsored jointly by the Navy Department Bureau of Ordnance and Harvard University. The book contains numerous photographs and drawings. The technical addresses covered eight sessions dealing with the general topics of "Existing Calculating Machines", "The Logic of Large Scale Calculating Machinery", "Storage Devices", "Numerical Methods and Suggested Problems for Solution", "Sequencing, Coding and Problem Preparation", "Input and Output Devices", "Conclusions and Open Discussion". The state of the art seems to have been well surveyed.

ROHN TRUELL

Physics in industry: The measurement of stress and strain in solids. Institute of Physics, London, 1948. x + 114 pp. $4.00.

This small book is based on the Proceedings of a Conference held in July 1946 by the Manchester and District Branch of the Institute of Physics. The topics covered include wire-resistance and other electrical strain gages and instrumentation, acoustic gages, two and three dimensional photoelasticity, and X-ray diffraction. Most of the papers are mainly summaries of the highlights of existing information and so are of more interest to British than to American workers who have the Proceedings of the Society for Experimental Stress Analysis and several books on some phases of the field available. However, there are a number of fundamental points concerning the physical behavior of wire-resistance gages which are raised and answered by Mr. Eric Jones in his very interesting and important contribution which should be read by all. A few of the other features which have not received much attention here are high frequency strain gages, the acoustic gage, and the precision setting of distance from film to specimen instead of using a reference metal in X-ray back reflection.

D. C. Drucker

Heat conduction with engineering and geological applications. By L. R. Ingersoll, O. J Zobel, and A. C. Ingersoll. McGraw-Hill Book Company, Inc., New York and London, 1948. xii + 278 pp. $4.00.

The subject of heat conduction is treated in two ways in various texts. As part of the general subject of heat transfer, heat conduction is treated from a practical point of view in various books on engineering applications of heat transfer. The mathematical theory, on the other hand, has been treated extensively and rigorously in several good texts. The intermediate region which gives sufficient mathematical background for a real understanding of the problems involved and yet a sufficient discussion of practical applications so as to make obvious the mode of application of the theory has been occupied by the book by Ingersoll and Zobel for many years.

The present work is a thorough revision of the older text with further engineering and geological applications. No attempt is made to give a rigorous mathematical foundation for the subject. On the other hand, several of the more advanced methods of solution of steady and non-steady heat flow problems are presented in a form which makes their ready use possible. Problems of increasing difficulty are treated through the book in the following order: Steady-state one-dimension, steady-state more than one dimension, periodic heat flow in one dimension, general non-steady heat flow in one dimension, and general non-steady heat flow in more than one dimension. A chapter is devoted to the classical treatment of the more complex problem of the two conducting media with phase change at the interface as in the formation of ice. Two final chapters deal with numerical and graphical methods for solving various types of heat conduction problems and a brief discussion of the methods of measuring the various thermal properties. Although the major applications are taken from geological problems, the book is a good one regardless of the field of application in which the reader is interested. The authors state as their aims, "... to develop the subject with special reference to the needs of the student who has neither time nor mathematical preparation to pursue the studies at greater length ..." and "... to point out the many applications of which the results are susceptible. .." The book succeeds admirably in living up to these stated purposes and is well worth consideration by anyone interested in applications of heat conduction.

H. W. Emmons

Applied mathematics for engineers and scientists. By S. A. Schelkunoff. D. Van Nostrand Co., Inc., Toronto, New York and London, 1948. xi + 472 pp. $6.50.

This latest addition to the rapidly increasing number of books on "Applied Mathematics for the Engineer, etc.", seems particularly appropriate for those scientists whose training has not been that of a mathematician. The more elementary material includes a presentation of the notions of differentiation,

integration, and the algebra of complex numbers and vectors. The former topics are not treated at great length but no knowledge of the calculus is taken for granted. Chapters are devoted to interpolation, solution of algebraic and transcendental equations, power series, and the introduction of exponential and related functions. For reference use a chapter on coordinate systems (listing the conventional operators in these systems) is included as well as chapters on Bessel functions, Fresnel integrals, and other transcendental functions. An extensive chapter on second order ordinary differential equations contains methods of solution ranging from the most elementary to those for obtaining asymptotic developments. The material on partial differential equations stresses the product-series solution representation. Conformal mapping (largely a discussion of certain explicit mappings), the use of the Laplace transform, and contour integration are treated, and the book ends with the derivation, from many physical problems, of the appropriate differential equations. Nearly all procedures (from integration to approximation techniques) are related to physical processes. Those analogs are usually electrical in character and the non-electrically trained may well prefer the mathematical method *per se*.

Each new topic is introduced in an extremely simple and clear manner at the minor expense, for example, of scattering bits of complex variable theory over several chapters. Much of the important material in the topics given above has been presented, and the book should be valuable to those interested in both introductory material and the conventional techniques applicable to many physical problems.

G. F. Carrier

Computation curves for compressible fluid problems. By. C. L. Dailey and F. C. Wood. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London. x + 33 pp. $2.00.

This set of computation curves is intended as a supplement to the text of Liepmann and Puckett "Introduction to Aerodynamics of a Compressible Fluid". Following a brief exposition concerning the equations used, there are three sets of charts which are frequently useful in speeding up computational work. The first set of graphs are plots of the conventional point functions of the Mach Number (e.g. Pressure/stagnation pressure v.s. Mach No., etc.). for example, (1) the shock angle, and (2) the stagnation pressure ratio, are plotted as functions of incoming Mach number and deflection angle. In section 3 the corresponding information on conical shocks is given. In the latter section, γ is taken as 1.405. In the others $\gamma = 1.4$.

G. F. Carrier

Tables of the Bessel functions of the first kind of orders forty through fifty-one. By The Staff Of The Computation Laboratory. Harvard University Press, Cambridge, 1948. 620 pp. $10.00.

The Bessel Functions $J_{40}(x)$, $J_{41}(x)$, $\cdots$, $J_{51}(x)$, are tabulated to ten decimal places. The argument varies in steps of .01 for $0 \leq x < 100$.

G. F. Carrier

Contributions to applied mechanics (Reissner Anniversary Volume). Edited by The Staff Of The Department Of Aeronautical Engineering and Applied Mechanics Of The Polytechnic Institute of Brooklyn. J. W. Edwards, Ann Arbor, Michigan, 1949. viii + 493 pp. $6.50.

This testimonial to the great contributions of Professor H. Reissner to aeronautical and structural engineering, applied mathematics, mechanics, and to physics contains excellent original and expository papers by many of his eminent colleagues and friends. Lack of space prevents the inclusion of even the

titles of all the papers and permits only the listing of the authors under the general classification of their subject. AERODYNAMICS: S. Bergman, R. Paul Harrington and Paul A. Libby, Henry G. Lew, D. P. Riabouchinsky, Walter Tollmien. DYNAMICS: Martin Goland, Paul Lieber and M. E. Hamilton, Rufus Oldenberger, D. Williams, S. W. Yuan and M. Morduchow. ELASTICITY AND STRUCTURES: L. H. Donnell, K. O. Friedrichs, R. Gran Olsson, Eric Reissner, A. Schleusner, George Schnadel, J. J. Stoker, N. J. Hoff, V. L. Salerno, Harold Liebowitz, Bruno A. Boley, Sebastian V. Nardo. ELECTRICITY: Ronald M. Foster, Reinhold Rüdenberg. MATHEMATICAL METHODS: Hilda Geiringer, R. Grammel, Alexander Weinstein. PLASTICITY: R. v. Mises, A. Nadai, Folke K. G. Odqvist. PROPULSION: Theodore von Kármán, Paul Torda.

D. C. Drucker

Tables of generalized sine- and cosine-integral functions: Part I. By The Staff Of The Computation Laboratory. Harvard University Press, Cambridge, 1949. vii + 462 pp. $10.00.

The twenty-eight introductory pages of this volume are concerned with the definitions of the functions to be tabulated, a discussion of the computation methods, the interpolation technique, a brief application list with bibliography, and tables of certain coefficients used in computing the tables which comprise the remainder of the book. Six functions are computed: They are $\int^{x}_{0} F^{n}(a, s)ds$ where the six integrands in question are $u^{-1} \sin u$, $u^{-1}(1 - \cos u)$, $u^{-1} \sin u \sin x$, $u^{-1} \sin u \cos x$, $u^{-1} \cos u \sin x$, $u^{-1} \cos u(1 - \cos x)$, with $u(x, a) = (x^2 + a^2)^{1/2}$.

These functions are tabulated in the range $0 \leq a \leq 1$ in steps of .01 for $0 \leq x \leq 1$ with increments $\Delta x = .01$. When a is a multiple of .05, they are also given for $1 \leq x \leq 2$ with $\Delta x = .02$ and $2 < x \leq 5$ with $\Delta x = .05$. For certain other values of a the tables for $1 < x < 2$ are also included. For $1 < a < 2$, the increments are in general larger but not uniform and values of the functions are available for $0 \leq x \leq 25$ when a is a multiple of .1.

G. F. Carrier

Tables of generalized sine- and cosine-integral functions: Part II. By The Staff Of The Computation Laboratory. Harvard University Press, Cambridge, 1949. 560 pp. $10.00.

This volume is a continuation of the tables described in the foregoing review. The range of a is $2 \leq a \leq 25$, $0 \leq x \leq 25$. The increments in a are: .05 for $a < 5$, .1 for $5 < a < 10$, .2 for $a > 10$. The increments in x are also: .05 for $x < 5$, .1 for $x < 10$, .2 for $x < 25$. However, the tables run to $x = 25$ only for intermittent values of a. For some values of a they terminate at $x = 5$, for others at $x = 10$.

G. F. Carrier

Vectorial mechanics. By E. A. Milne. Interscience Publishers Inc., New York, 1948. xiii + 382 pp. $7.50.

This text furnishes a thorough and unified treatment of mechanics by Gibbs' vector and dyad analysis. The author's decision to use only this tool, limits the scope of the text to some extent. Thus, the following topics are omitted: (1) the Lagrange generalized coordinates; (2) variational method; (3) the integration of the rotational equations of motion of a rigid body in terms of the Eulerian angles, which are defined in the text. Further, the author's very complete introduction to Gibbs' vector analysis leads to the use of such uncommon terms as, "tensor of a vector", "vector of a tensor". However, these disadvantages are more than compensated for by the author's clear and unified treatment of topics.

In particular, the following discussions should be of interest to the reader: (1) a vector technique for solving linear vector differential equations with constant coefficients; (2) an elegant vector treatment of Euler's theorem on rotations about a point in a rigid body; (3) a simple and interesting vector decomposition proof of the triple vector expansion; (4) a very general tensor formulation of the Gauss and Stokes integral theorems. The author's treatment of the last mentioned topic is facilitated by introducing the index notation and the basic concepts of tensor algebra.

The following topics are discussed in the order listed; (1) vector and tensor analysis; (2) systems of line vectors with applications to statics and small displacements of a rigid body; (3) kinematics and dynamics of a particle and a rigid body. The material of the text is fully illustrated by numerous problems. Many of these problems are of intrinsic interest and are completely solved by the author. In the general theory and the problems, the author's methods illustrate the power of the vector approach to mechanics.

In short, the author has written an excellent text, which even the expert may profitably peruse.

N. Coburn

Advances in applied mechanics. Edited by Richard von Mises and Theodore von Kármán. Academic Press Inc., New York, 1948. viii + 293 pp. $6.80.

This book is the first of a series intended to present the results of current research in various fields of Applied Mechanics in the form of collections of expository monographs and summaries. The parts of this volume are described separately.

Hugh L. Dryden, "Recent advances in the mechanics of boundary layer flow". The subject is discussed with particular reference to the problem of the stability of a laminar boundary layer and to the nature of turbulent flow in boundary layers. The nature of the laminar boundary layer and the effects of boundary layer suction are also discussed. An extensive bibliography is presented.

N. Minorsky, "Modern trends in non-linear mechanics". After a brief history of the development of non-linear mechanics the modern approach is presented. The first section deals with topological methods and gives the results obtained and the inherent limitations of this attack. The second section treats the analytical methods of Poincaré, van der Pol, Kryloff and Bogoliuboff, and others. The third section deals with non-linear resonance and associated phenomena.

C. B. Biezeno, "Survey of papers on elasticity published in Holland 1940-1946". As indicated by the title this contribution is in the nature of a review. The works discussed are diverse and lie in all of the subject fields of elasticity.

J. M. Burgers, "A mathematical model illustrating the theory of turbulence". A simplified mathematical model is devised with the intention of simulating the problem of turbulence in an incompressible fluid. The model does not have the same complicated geometrical character as the turbulence problem but retains the essential non-linearity and, in analogue, the characteristics of energy transfer and dissipation. From the mathematical treatment results analogous to those of the isotropic theory of turbulence are obtained and suggestions are made as to other characteristics of turbulent fluid motion.

H. Geiringer, "On numerical methods in wave interaction problems". The problem of calculating the unsteady one-dimensional flow fields in a perfect gas is investigated. The principal treatment is based on the Hugoniot shock relations; the approach of J. von Neumann is also presented.

R. von Mises and M. Schiffer, "On Bergman's integration method in two-dimensional compressible fluid flow". Part I presents the general theory of S. Bergman based upon the equation for the stream function in the hodograph plane. Part II presents the particular treatment where a simplified pressure-density relation is assumed.

This book and future volumes of the same series make possible the publication of papers of expository, review, or developmental nature in monograph length. This is particularly valuable because of the present dearth of monograph journals and the shortage of space in the regular journals in applied mechanics and mathematics. The future of the venture will, of course, depend upon demand and upon the quality of profferred contributions.

To quote from the preface—"The present volume might be considered as indicative of the topics

to be dealt with in future issues and as an example for the diversified kinds of approach we wish to cultivate. In both respects, however, the Editors reserve a certain freedom of choice. Suggestions are invited and the offering of contributions will be appreciated."

WALLACE D. HAYES

Calcolo tensoriale e applicazioni. By Bruno Finzi and Maria Pastori. Nicola Zanichelli Editore, Bologna, 1949. vii + 427 pp. Lire 2000.

The authors' purpose, as stated in the introduction, is to present the concepts and principal methods of the tensor calculus, in order to facilitate its application by mathematicians, physicists, and engineers. The reader's minimum mathematical background is presupposed to be a (first course) knowledge of the differential and integral calculus. The exposition is careful and orderly, proceeding by easy stages; and the authors have indeed achieved their aim.

The book consists of an introduction, ten chapters, and a bibliography. The introduction is in the nature of an orientation; the reader is "briefed" on the ground to be covered and acquainted with the concepts that will occur again and again, e.g., vectors, tensors, contravariant superscripts, covariant subscripts, etc. A pleasant feature of the book is the table of principal formulas ("formule notevoli") which follows every chapter. The first three chapters deal with the tensor calculus proper: vector fields, tensors and tensor algebra, and "omografie vettoriali" (linear vector functions). The next four chapters are concerned with applications to differential geometry: tensor fields in Euclidean and non-Euclidean spaces, surface geometry and Riemannian geometry. The last three chapters contain the applications to mathematical physics referred to in the first paragraph of the introduction. The authors have succeeded in compressing an unusually large amount of material into comparatively few pages. The chapter headings are: VIII Mechanics of Deformable Continua, IX Electromagnetic Theory, and X Relativity Theory. The book concludes with a bibliography of representative works on various phases of the subject.

J. B. DIAZ

Supersonic flow and shock waves. By R. Courant and K. O. Friedrichs. Interscience Publishers, Inc., New York and London, 1948. xvi + 464 pp. $7.00.

This book is primarily concerned with the hyperbolic problems associated with the behavior of continua. Except for brief remarks about shallow water waves and discontinuous waves in plastic media, attention is confined to gas flows. The opening chapter provides the fundamental notions (thermodynamic concepts, and basic mechanical laws) which form the basis of the analysis of these phenomena. This is followed by a chapter on the mathematics of hyperbolic systems of second order. Unsteady one-dimensional problems are then considered; shock interaction, detonation, and plastic waves are discussed in this section. Chapter IV deals with steady plane flow; it utilizes the hodograph method, to present certain isentropic flows, and contains the analysis of shock interactions, and treats the flow past obstacles, etc. by the perturbation method. The book ends with briefer discussions of nozzle flows, jets, axially symmetric flows, spherical waves and certain conical flow problems.

The book seems well adapted for the presentation to students of that portion of compressible fluid theory which is essentially hyperbolic. It is obviously not intended as a self contained text on the general field of compressible flow theory.

G. F. CARRIER